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ON THE DETERMINACY OF GAMES ON ORDINALS

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Let λ be an ordinal and $A \subseteq \lambda^\omega \times \lambda^\omega$. As usual we associate with it the game $G(A;\lambda)$:

$$\begin{array}{lll}
\mathrm{I} & \mathrm{II} & \text{I and II alternatively} \\
\xi_0 & & \\
 & \eta_0 & \text{play } \xi_0, \eta_0, \xi_1, \eta_1, \dots \text{ from } \lambda; \\
\xi_1 & & \text{I wins iff } (\vec{\xi}, \vec{\eta}) \in A. \\
\vdots & \eta_1 & \\
\vec{\xi} & \vec{\eta} &
\end{array}$$

When $\lambda < \Theta =$ first ordinal (other than 0) not the surjective image of ω^ω, we can find a norm $\phi : \omega^\omega \twoheadrightarrow \lambda$ and then consider the coded version of the above game, denoted by $G_\phi^*(A;\omega^\omega)$:

$$\begin{array}{lll}
\mathrm{I} & \mathrm{II} & \text{I and II alternatively} \\
x_0 & & \\
 & y_0 & \text{play } x_0, y_0, x_1, y_1, \dots \text{ in } \omega^\omega; \\
x_1 & & \text{I wins iff } ((\phi(x_0), \phi(x_1), \dots), \\
\vdots & y_1 & (\phi(y_0), \phi(y_1), \dots)) \in A. \\
\vec{x} & \vec{y} &
\end{array}$$

From the Axiom of Choice it follows trivially that the two games G and G^* are equivalent.

We develop in this paper (Section 5) a method for simulating the games G^* as

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above, for certain ϕ's, by games on ω and therefore we show that the standard definable determinacy hypotheses for games on ω imply also the determinacy of definable games on certain (uncountable) ordinals and their coded versions. This method is applicable to a wide range of definable norms ϕ, certainly including the projective ones. (In terms of ordinals, this means that $\omega_1 = \boldsymbol{\delta}^1_1, \boldsymbol{\delta}^1_2, \boldsymbol{\delta}^1_3, \ldots$ are all covered by it.) It is an improvement of a basic technique devised by Harrington [2], who first proved such results on ordinal game determinacy for certain $\boldsymbol{\Delta}^1_3$-norms (of length u_ω = the ωth uniform indiscernible).

One special (almost degenerate) case of ordinal games deserves particular attention, because of its wide applicability. We are given a $P \subseteq \lambda \times \omega^\omega$ and we consider the game:

I	II	
		I plays $\xi < \lambda$, II then
ξ		
	$\alpha(0)$	plays successively
—		
	$\alpha(1)$	$\alpha(0), \alpha(1), \ldots \in \omega$; II wins
—		
	$\vdots$	iff $P(\xi, \alpha)$.
	α	

We study in Section 1, what are essentially coded versions of these 'one-step' ordinal games and by applying our general technique in this special context, we derive a useful uniformization theorem for relations of the form $R(w, x)$ which are invariant (on w) with respect to certain norms ϕ (i.e. $R(w, x) \wedge \phi(w) = \phi(v) \Rightarrow R(v, x)$). This allows us also to present most of the key ideas of our general method in a simplified framework. Then after discussing in Section 2 the connections of this uniformization theorem with the earlier results of Moschovakis [13], we give in Section 3 some of its applications, especially to the problem of ordinal quantification in the analytical hierarchy. We show, for example, that, assuming PD, Σ^1_{n+1} relations are closed under quantification over Δ^1_n ordinals. (An ordinal is Δ^1_n if it is the rank of a Δ^1_n prewellordering of ω^ω-such ordinals are in general uncountable.) Finally, using heavily the results in Sections 2, 3 we prove in Section 4 Moschovakis' conjecture, that the set of reals in $L[T^{2n+1}]$ is precisely $\mathscr{C}_{2n+2}$, the largest countable Σ^1_{2n+2} set (assuming $\mathrm{Det}(L[\omega^\omega])$).

After the ordinal determinacy results of Section 5 that we mentioned in the beginning, we collect in Section 6 some of their further applications and in Section 7 we give, using ideas from Sections 5, 6 now, a proof of the fact that $\omega^\omega \cap L(T^{2n+1}] = \mathscr{C}_{2n+2}$ *in PD only*. In Section 8 we briefly sketch a development of *projective set theory*, i.e. a 'projective analog' of certain aspects of the standard set theory of ordinals, cardinals etc. which can be worked out in PD only and is made possible mainly by the results in Section 3. One of its applications is in providing a general method for replacing uses of AD by PD, in proofs of results about projective sets. Finally, in Section 9 we discuss some open problems.

In concluding this introduction, we would like to thank Ramez Sami for many illuminating discussions on the matters presented in this paper and especially for steering our thinking towards the uniformization formulation of the results in Section 1.

0. Preliminaries

Our notation and terminology in this paper will be standard, following in most instances that of Moschovakis [12]. Our *basic spaces* will be ω, $\mathcal{R} = \omega^\omega$ (the *reals*) and $\mathcal{C} = 2^\omega$. *Product spaces* are of the form $\mathcal{X} = X_1 \times \cdots \times X_k$, where each X_i is a basic space. Members of these product spaces are called *points* and subsets of them are called *pointsets*. We also view them as relations and write interchangeably: $R(x) \Leftrightarrow x \in R$, for $R \subseteq \mathcal{X}$. A *pointclass* is a collection of pointsets. *Spector pointclasses* appear very often in this paper and we refer the reader to [12] for the definition and their basic properties, which we use throughout without explicit mentioning. More generally, standard results from descriptive set theory which we use without reference can be always found in [12].

Some notational conventions: Letters a, b, c vary always over 2^ω: α, β, γ, δ, ε over ω^ω; e, i, j, k, l, m, n over ω and ξ, η, ϑ, κ, λ, μ, ρ over ordinals. The notation $f: X \twoheadrightarrow Y$ signifies an *onto* map, while $g: X \rightarrowtail Y$ an *injective* one.

Also, $P(A)$ denotes the power set of A, Θ the supremum of the ranks of prewellorderings of ω^ω and for any pointclass Γ, $\mathrm{Det}(\Gamma)$ abbreviates the determinacy of all $A \in \Gamma$. Finally, by $\mathrm{IND}(\mathbb{R})$ (resp. $\mathrm{HYP}(\mathbb{R})$) we denote the pointclass of all absolutely inductive (resp. absolutely hyperprojective) pointsets. More generally, $\mathrm{IND}(\mathbb{R}, R)$, $\mathrm{HYP}(\mathbb{R}, R)$ refer to the corresponding relativized notions, for each $R \subseteq \omega^\omega$. ($\mathbb{R}$ stands here for the structure of analysis.)

All the results in this paper are proved in ZF + DC, with all further hypotheses stated explicitly.

PART I: One-step ordinal games

1. A uniformization theorem for invariant pointsets

1.1. Let κ be an ordinal $< \Theta$. Let also $P(\xi, x)$ be a predicate, where ξ varies over ordinals $< \kappa$ and x over some product space $\mathcal{X}$. Very often one is faced with the problem of knowing that $\forall \xi\, \exists x\, P(\xi, x)$ and searching for a definable uniformizing function $F: \kappa \to \mathcal{X}$, i.e. a function such that $\forall \xi\, P(\xi, F(\xi))$. This is actually an ill-posed formulation of the problem, which has clearly a negative answer. Just take $\phi: W \twoheadrightarrow \kappa$ to be a norm on a set $W \subseteq \omega^\omega$ onto κ, i.e. a coding system of ordinals $<\kappa$, and let $P(\xi, x) \Leftrightarrow \phi(x) = \xi$. A definable uniformizing function F for this P would obviously choose exactly one code for each $\xi < \kappa$, thereby producing a definable wellordering of a subset of ω^ω of order type κ, clearly an impossibility if κ is uncountable, according to current beliefs on the nature of the continuum, which do not allow the existence of *uncountable definable* wellorderings of sets of reals.

So let us turn now to the correct formulation of the problem. Let, in the above notation, $P^*(w, x) \Leftrightarrow w \in W \wedge P(\phi(w), x)$ be the *coded version* of P. Motivated by the preceding discussion, we might ask instead about the existence of a definable uniformizing function $G: W \to \mathscr{X}$ for the pointset P^*, at least when the norm $\phi: W \twoheadrightarrow \kappa$ and P^* are definable. This is clearly a well-posed problem, apparently a special case of the general Uniformization Problem for pointsets, for which many successful solutions have been found. In practice however, the general answers provided by the uniformization theory, in the context of various hypotheses of definable determinacy, are not of much use in our specialized situation and the problems that arise in it, as it will be seen by various examples to be developed as we proceed. The main reason is this: The pointset P^* as above could be of very high complexity, according to some definability measure, so that the general theory can provide us at best only with also a fairly complex uniformizing function G, if one at all, while the needs of our problem require a much simpler such G. For instance, in a given situation, P^* could be $\boldsymbol{\Pi}^1_9$, whereby we could produce in general a G with $\boldsymbol{\Pi}^1_9$ graph, while we actually need one with $\boldsymbol{\Delta}^1_5$ graph. Or, in the limit but quite common case, P^* could be an essentially arbitrary definable pointset for which, on general grounds, no definable uniformization is known (or ought) to exist.

Thus the results we will be discussing in this section, belong to a special chapter of uniformization theory, which deals with the particular uniformization problem of pointsets of the form P^* as above and the particular solutions that can be obtained in this case. As it will turn out, the definitional complexity of P^* is irrelevant for this problem. What really matters is the complexity of the norm $\phi: W \twoheadrightarrow \kappa$ (in terms of its associated prewellordering $w \leq_\phi v \Leftrightarrow w, v \in W \wedge \phi(w) \leq \phi(v)$). However, the complexity of P^* affects the amount of determinacy hypotheses used in each case.

1.2. Let us turn now to specifics. In order to make clearer the essence of some arguments, but also to achieve somewhat wider applicability, we shall work in a more general context than before, which we will immediately explain.

1.2.1. Definition. Let $\sim$ be an equivalence relation with field $W \subseteq \omega^\omega$. We shall denote by $|w|_\sim$, or just $|w|$ when there is no danger of confusion, the $\sim$-equivalence class of $w \in W$. Let also $K_\sim \equiv K$ be the set of equivalence classes of $\sim$. One of our standard examples in the sequel will be the equivalence relation $\sim_\phi$ induced by a norm $\phi: W \twoheadrightarrow \kappa$, where $W \subseteq \omega^\omega$, i.e. $w \sim_\phi v \Leftrightarrow w, v \in W \wedge \phi(w) =$

$\phi(v)$. In this case we can identify, with no harm, $|w|$ with $\phi(w)$, and K with κ. We shall call, following Sami's terminology, $\sim$ *coarse* if $\sim$ admits no perfect set of pairwise $\sim$-inequivalent elements. (Perfect means always nonempty in this paper.) Clearly $\sim_\phi$ as above is coarse, provided $\leqslant_\phi$ belongs to a reasonable class Γ, for which $\mathrm{Det}(\Gamma)$ holds.

If now $R \subseteq K \times \mathscr{X}$, where $\mathscr{X}$ is a product space, let $R^*_\sim \equiv R^*$ given by

$$R^*(w, x) \Leftrightarrow w \in W \wedge R(|w|, x),$$

be the *coded version* of R, thinking of course of w as a *code* of $|w|$. Clearly R^* is a $\sim$-*invariant* (on w) pointset, i.e.

$$R^*(w, x) \wedge v \sim w \Rightarrow R^*(v, x).$$

Conversely, every such pointset S gives rise to a relation $S_* \subseteq K \times \mathscr{X}$, defined by

$$S_*(\xi, x) \Leftrightarrow \exists w \in W(S(w, x) \wedge |w| = \xi).$$

We shall state and prove below a uniformization theorem for invariant pointsets, under certain conditions. We shall need though first some convenient terminology.

1.2.2. Definition. Let Γ be a Spector pointclass. We call Γ *category-adequate* iff the following conditions hold:

(i) Every pointset in $\boldsymbol{\Gamma}$ has the property of Baire.

(ii) For each $Q \subseteq \mathscr{X} \times \mathscr{Y}$ in Γ, the pointset $P \subseteq \mathscr{X}$ given by $P(x) \Leftrightarrow \{y: Q(x, y)\}$ is not meager, is also in Γ.

(iii) If $A \subseteq \mathscr{Y} \times 2^\omega$ is in $\Gamma(x)$, for some point x, and if for each $y \in \mathscr{Y}$, $A_y = \{a: A(y, a)\}$ is not meager, there is $T \subseteq \mathscr{Y} \times \omega$ in $\Delta(x)$, such that for all $y \in \mathscr{Y}$, T_y is a perfect binary tree and for all $y \in \mathscr{Y}$, $[T_y] \subseteq A_y$. (A binary tree is a tree on $2 = \{0, 1\}$ and we identify above $2^{<\omega}$ with ω.)

The standard examples of category-adequate Spector pointclasses are given in the following lemma.

1.2.3. Lemma. (i) *Assume* $\mathrm{Det}(\boldsymbol{\Delta}^1_{2n})$. *Then* Π^1_{2n+1} *is category-adequate. If also* $\mathrm{Det}(\boldsymbol{\Sigma}^1_{2n+1})$ *holds,* Σ^1_{2n+2} *is category adequate.*

(ii) *Assume* $\mathrm{Det}(\mathbf{IND}(\mathbb{R}))$. *Then* $\mathrm{IND}(\mathbb{R})$ *is category-adequate.*

It should be pointed out here, that (modulo relativizations) $\mathrm{IND}(\mathbb{R})$ is the largest known category adequate pointclass. This delineates the current limits of applicability of the results in this paper.

We can state now the main result of this section.

1.2.4. Theorem. *Let Γ be a category-adequate Spector pointclass and $\sim$ a coarse equivalence relation on $W \subseteq \omega^\omega$ with $\sim \in \boldsymbol{\Gamma}$. Given any $\sim$-invariant $R(w, x)$ such* *that* $\forall w \in W \exists x R(w, x)$, *there is a uniformizing function* $G: W \to \mathscr{X}$ (*i.e.* $\forall w \in W R(w, G(w))$) *which is* $\boldsymbol{\Gamma}$*-measurable* (*i.e.* $\{w \in W: G(w) \in N\} \in \boldsymbol{\Gamma}$, *for each open set* $N \subseteq \mathscr{X}$), *granting* $\mathrm{Det}(\langle \Gamma; \neg W, R \rangle)$, *where* $\langle \Gamma; R_1 \cdots R_n \rangle$ *is the smallest pointclass containing* Γ, $R_1 \cdots R_n$ *and closed under* $\wedge$, $\vee$ *and the bounded quantification* $\exists x \in \Delta(y)$.

Let us emphasize once again, that the complexity of the uniformizing function G has nothing to do with the complexity of the invariant relation R, but only with that of the equivalence relation $\sim$.

1.3. Proof. To motivate some key ideas in the proof, let us assume for the moment that $\sim = \sim_\phi$, where $\phi: W \twoheadrightarrow \kappa$ is a norm, so that $|w| = \phi(w)$. If $P = R_*$, we know that

$$\forall \xi \, \exists x \, P(\xi, x). \tag{1}$$

To prove the desired conclusion of the theorem, the first attempt is to use a Solovay-type game as follows:

I	II	II wins iff
w	x	$w \in W \Rightarrow R(w, x)$.

If II has a winning strategy in this game, we have actually a continuous uniformizing function for R. But how can we exclude the possibility that I has a winning strategy, say σ? If $A = \{\sigma(x): x \in \mathscr{X}\}$, clearly $A \subseteq W$ is a $\boldsymbol{\Sigma}^1_1$ set. If now ϕ had the *boundedness property* i.e. every $\boldsymbol{\Sigma}^1_1$ subset of W was bounded below κ and if moreover P had the following extra property

$$\forall \xi \, \exists x \, \forall \eta < \xi \, P(\eta, x), \tag{2}$$

then we would get our contradiction, as usual: Let first ξ be such that $\forall w \in A$, $|w| < \xi$. Then let x be such that $\forall \eta < \xi \, P(\eta, x)$. If II plays this x, he beats σ.

Unfortunately, in the very general context of our theorem (still for the case $\sim = \sim_\phi$), the boundedness property will not be many times true (usually in fact $W = \omega^\omega$), but, even more seriously, condition (2) above will also in general fail, being much stronger than (1).

Let us return now to the case of an arbitrary coarse $\sim$, as the special case $\sim = \sim_\phi$ would serve no further illustrative purpose now. Keeping the same game as above, a further attempt to save the argument would be to make sure that all moves $\sigma(x)$ of I fall in a single equivalence class of $\sim$. This is, however, unlikely to happen for *all* $x \in \mathscr{X}$. But perhaps a large enough set of x's suffices. The coarseness of $\sim$ indeed guarantees that there is a perfect set $Q \subseteq \mathscr{X}$ so that $\sigma''Q$ is contained in a single equivalence class of $\sim$. (We do not give the argument here as it is essentially included in the proof of Sublemma 1.4.3 below). Say $\sigma''Q \subseteq |w_0|$. Using that $\forall w \in W \exists x \, R(w, x)$, choose now x_0 with $R(w_0, x_0)$. If we could find such a x_0 in Q, the contradiction would be immediate. But clearly there is no reason for this to happen. On the other hand, if we take, as we can do without any loss of generality, $\mathscr{X}$ to be 2^ω and we let $Q=[T]$, where T is a perfect binary tree, there is a canonical homeomorphism $h_T:[T]\twoheadrightarrow 2^\omega$ and a (unique) $y_0\in[T]$ with $h_T(y_0)=x_0$. So perhaps instead of playing x directly, player II could play a perfect binary tree T and a path $y\in[T]$ and win iff $R(w, h_T(y))$, provided of course $w\in W$. With this new variation of the game, if σ is a winning strategy for I, $\sigma(T, y)\in W$ for all T, y and for each fixed T the argument before produces a T^*, with the equivalence class of $\sigma(T, y)$ the same for all $y\in[T^*]$. But there is no guarantee that there is a T with $T=T^*$, so that we still cannot obtain a contradiction. We can however isolate now what is needed to carry out this argument. First of all, the requirement that II plays directly the tree T is too stringent and leaves little hope for obtaining a 'fixed point' $T=T^*$. So let us be a bit more flexible and consider the following concept.

1.3.1 A *coding system* for perfect binary trees is a set $P\subseteq 2^\omega$ together with a map $\mathscr{P}:P\twoheadrightarrow\{T\colon T \text{ is a perfect binary tree}\}$. If $a\in P$, we put $\mathscr{P}(a)\equiv\mathscr{P}_a=$ the tree coded by a. If $c\in[\mathscr{P}_a]$, let $a^*(c)=$ the real in 2^ω corresponding to c under the canonical homeomorphism $a^*:[\mathscr{P}_a]\twoheadrightarrow 2^\omega$. Finally, call $\langle a, c\rangle$ *good* (relative to $\mathscr{P}$) if $a\in P\wedge c\in[\mathscr{P}_a]$. (Here $\langle\,\rangle$ is the standard recursive homeomorphism of $2^\omega\times 2^\omega$ with 2^ω.)

With this more general concept at hand, the game to be played is now as follows, assuming without loss of generality that $\mathscr{X}=2^\omega$:

I	II	II wins iff
w	$\langle a, c\rangle$	$w\in W\Rightarrow[\langle a, c\rangle \text{ is good}\wedge R(w, a^*(c))]$

If σ is a winning strategy for I in this game, put

$$H(a, c)=\sigma(\langle a, c\rangle).$$

Then clearly $H:2^\omega\times 2^\omega\to\omega^\omega$, range$(H)\subseteq W$ and H is continuous. The crucial 'fixedpoint' property that suffices to produce the desired refutation of this possibility (of I having a winning strategy) is now the following:

$(*)$ If $H:2^\omega\times 2^\omega\to\omega^\omega$ is $\boldsymbol{\Gamma}$-measurable and range$(H)\subseteq W$, then there is (effectively in a $\boldsymbol{\Gamma}$-code of H and $\sim$) $a\in P$ and $\xi\in K$ such that $\forall c\in[\mathscr{P}_a](|H(a, c)|=\xi)$.

(This is stated in a stronger form with '$\boldsymbol{\Gamma}$-measurable' instead of 'continuous', as we have in mind further applications of this concept in the second part of the paper.)

Granting that $(*)$ holds, let us repeat the argument that I cannot have a winning strategy: Let H be as above, and by $(*)$ find $a\in P$ so that $|H(a, c)|$ is the same, say $|w_0|$, for all $c\in[\mathscr{P}_a]$. Pick $x\in 2^\omega$ with $R(w_0, x)$. Pick the $c\in[\mathscr{P}_a]$, with $a^*(c)=x$. Then $R(w_0, a^*(c))$ and as $H(a, c)=\sigma(\langle a, c\rangle)\sim w_0$, we have $R(\sigma(\langle a, c\rangle), a^*(c))$ and $\langle a, c\rangle$ is good, so II won, while I played following his winning strategy σ, a contradiction.

Thus, granting the determinacy of this game, II has a winning strategy τ. Let $G: W \to 2^\omega$ be given by

$$G(w) = a^*(c), \quad \text{where } \langle a, c\rangle = \tau(w).$$

In order to assure that G is $\boldsymbol{\Gamma}$-measurable, the following condition on the coding system, is sufficient:

(**) $P \in \Gamma$ and for $a \in P$ the relation

$$[\mathcal{P}(s, a) \Leftrightarrow s \in \mathcal{P}_a]$$

is in Δ, i.e. there are relations A, B in Γ, $\neg\Gamma$ respectively, with the property that for $a \in P$, $\mathcal{P}(s, a) \Leftrightarrow A(s, a) \Leftrightarrow B(s, a)$.

Because, if (**) holds, we have

$$G(w)(n) = i \Leftrightarrow w \in W \wedge \exists\langle a, c\rangle \in \Delta(\tau, w)[\tau(w) = \langle a, c\rangle \wedge a^*(c)(n) = i]$$

which is clearly in $\Gamma(\tau)$. Moreover, granting (**), it is also clear that the payoff set for player II is in $\langle \Gamma; \neg W, R\rangle$, since it can be expressed as follows:

$$w \notin W \vee [a \in P \wedge c \in [\mathcal{P}_a] \wedge \exists x \in \Delta(\langle a, c\rangle)(a^*(c) = x \wedge R(w, x))]$$

1.4. Thus the completion of the proof has been reduced to finding a coding system satisfying conditions (*), (**). As we shall refer to them again in the second part of this paper let us introduce the following terminology.

1.4.1. Definition. Let $\sim$ be an equivalence relation on $W \subseteq \omega^\omega$, $\mathcal{P}$ a coding system for perfect binary trees and Γ a Spector pointclass. We call $(\sim, \mathcal{P}, \Gamma)$ *nice* iff conditions (*), (**) of 1.3. hold.

Thus we have only to prove:

1.4.2. Lemma. *If $\sim$ is a coarse equivalence relation on $W \subseteq \omega^\omega$, Γ a category-adequate Spector pointclass and $\sim \in \boldsymbol{\Gamma}$, then there is a coding system for perfect binary trees $\mathcal{P}$ such that $(\sim, \mathcal{P}, \Gamma)$ is nice.*

Proof. The argument is a combination of an effective category argument — using all the properties of a category-adequate Spector pointclass — and the Recursion theorem for the generalized recursion theory associated with the Spector pointclass. Let us first isolate in a sublemma the effective category result that we will need.

1.4.3. Sublemma. *Let Γ be a category-adequate Spector pointclass, $\sim$ a coarse equivalence relation with field $W \subseteq \omega^\omega$ and $h: \omega \times 2^\omega \to \omega^\omega$ a function with range included in W. If x is a point such that $\sim \in \Gamma(x)$ and h is $\Gamma(x)$-recursive (i.e. the relation $h(e, a)(n) = m$ is in $\Gamma(x)$), then there is a $\Delta(x)$ relation $T(e, s)$ such that for each $e \in \omega$, $T_e = \{s: T(e, s)\}$ is a perfect binary tree and $h_e''[T_e]$ falls in a single equivalence class of $\sim$, where $h_e(a) = h(e, a)$.*

Proof. Consider the following equivalence relation on 2^ω, for each $e \in \omega$

$$a \sim_e b \Leftrightarrow h_e(a) \sim h_e(b).$$

If $\sim_e$ (as a subset of $2^\omega \times 2^\omega$) is meager, it is easy to produce (combining the proof of the Baire Category theorem with a routine splitting argument) a perfect set Q of pairwise $\sim_e$-inequivalent elements. Then $h_e \upharpoonright Q$ is 1–1 and as it has the property of Baire (being Γ-measurable), we can shrink Q down to a perfect set $S \subseteq Q$ with $h_e \upharpoonright S$ continuous. Then $h_e''S$ is a perfect set of pairwise $\sim$-inequivalent reals, a contradiction. So each $\sim_e$ is nonmeager and thus, by the Kuratowski–Ulam theorem, one of its equivalence classes is not meager. Therefore $B_e = \{a \in 2^\omega : |a|_{\sim_e}$ is not meager$\}$ is a nonmeager set in $\Gamma(x)$, by (ii) of Definition 1.2.2. Then, by (iii) of Definition 1.2.2 we can find a_e in $\Delta(x)$, with $a_e \in B_e$, i.e. $|a_e|_{\sim_e}$ not meager. So

$$\forall e\, \exists a \in \Delta(x)[|a|_{\sim_e} \text{ is not meager}].$$

The condition in square brackets is in $\Gamma(x)$, so by standard arguments on Spector pointclasses, we can find a $\Delta(x)$-recursive function $f:\omega \to 2^\omega$, so that letting $f(e) = \bar{a}_e$, we have $\forall e[|\bar{a}_e|_{\sim_e}$ is not meager]. Put

$$A(e, a) \Leftrightarrow a \sim_e \bar{a}_e.$$

Then $A \in \Gamma(x)$ and for each $e \in \omega$, A_e is not meager, so by (iii) of Definition 1.2.2 again, there is $T(e, s)$ in $\Delta(x)$ with T_e a perfect binary tree and $[T_e] \subseteq |\bar{a}_e|_{\sim_e}$ for each $e \in \omega$. Then $h_e''[T_e]$ falls in a single equivalence class of $\sim$ and we are done.

Let us recall now some general facts about the generalized recursion theory associated with the Spector pointclass Γ:

For each point x, there is a partial function $\phi_k^x(e, \vec{n})$ with graph in $\Gamma(x)$, which is universal for the partial functions from ω^k to ω which have graphs in $\Gamma(x)$. We shall write for convenience

$$[e]_k^x(\vec{n}) = \phi_k^x(e, \vec{n}),$$

so that $[e]_k^x$ is the eth partial function from ω^k to ω with graph in $\Gamma(x)$. Moreover the relation,

$$U_k(e, x, \vec{n}, m) \Leftrightarrow [e]_k^x(\vec{n}) = m$$

is in Γ, for each fixed k. The 's-m-n theorem' holds, so that there are primitive recursive functions s with the property that

$$[e]^x(\vec{n}, \vec{m}) = [s(e, \vec{n})]^x(\vec{m})$$

(we skip the obvious embellishments from s and the square brackets, when there is no danger of confusion). Finally, as usual, the Recursion theorem goes through in its various equivalent forms, of which we shall make use of the following: If $g:\omega \to \omega$ is total recursive, there is $e^* \in \omega$ such that $[e^*]^x = [g(e^*)]^x$.

We can define now our coding system $(F, \mathscr{P})$ with the desired properties: Put

$$P = \{\langle e, b\rangle \colon b \in 2^\omega \wedge e \in \omega \wedge [e]^b : \omega \to \omega \text{ is total and is the characteristic function of a perfect binary tree, denoted by } T_{\langle e, b\rangle}\},$$

and for $a \in P$ let

$$\mathscr{P}_a = T_a.$$

Here

$$\langle e, b\rangle = \underbrace{(0, \ldots, 0)}_{e+1}{}^\frown 1^\frown b.$$

Clearly, (**) of Definition 1.4.1 is immediately satisfied. We now verify (*): Let $H: 2^\omega \times 2^\omega \to \omega^\omega$ be $\boldsymbol{\Gamma}$-measurable with range$(H) \subseteq W$. As $\sim \in \boldsymbol{\Gamma}$, find $b \in 2^\omega$ such that $\sim \in \Gamma(b)$ and also H is $\Gamma(b)$-recursive. We shall find an $e^* \in \omega$ such that $a = \langle e^*, b\rangle \in P$ and $\forall c \in [\mathscr{P}_a]$, $|H(a, c)| = \xi$, for some fixed $\xi \in K$. For that let

$$h(e, c) = H(\langle e, b\rangle, c).$$

Clearly, $h: \omega \times 2^\omega \to \omega^\omega$ and range$(h) \subseteq W$, while h is $\Gamma(b)$-recursive, so by sublemma 1.4.3, for each $e \in \omega$ there is $\xi_e \in K$ and T_e, a $\Delta(b)$ perfect binary tree, with $|h_e(c)| = \xi_e$ for all $c \in [T_e]$. Moreover, also by Sublemma 1.4.3, we have that actually the relation $T(e, s) \Leftrightarrow s \in T_e$, is in $\Delta(b)$.

Now by the s-m-n theorem, there is a total recursive function $g: \omega \to \omega$ such that for all e, $[g(e)]^b$ is total and is the characteristic function of T_e. Then, by the Recursion theorem, let e^* be such that $[g(e^*)]^b = [e^*]^b$. If we take $a = \langle e^*, b\rangle$, then $a \in P$, $\mathscr{P}_a = T_{e^*}$ and for all $c \in [\mathscr{P}_a]$ we have that $|H(a, c)| = |H(\langle e^*, b\rangle, c)| = h(e^*, c) = \xi_{e^*}$, so we are done.

The proof of the theorem is now complete.

1.5. Let us put down as an illustration some particular instances of Theorem 1.2.4, making use of Lemma 1.2.3.

1.5.1. Corollary. *Let $\sim$ be a coarse $\boldsymbol{\Delta}^1_{2n+1}$ equivalence relation on $W \subseteq \omega^\omega$ and let $R(w, x)$ be $\sim$-invariant with $\forall w \in W\, \exists x\, R(w, x)$. Then there is a $\boldsymbol{\Delta}^1_{2n+1}$ function G uniformizing R. We assume here* $\mathrm{Det}(\langle \Pi^1_{2n+1}; \neg W, R\rangle)$. *Similarly, with $\boldsymbol{\Delta}^1_{2n+1}$, $\boldsymbol{\Pi}^1_{2n+1}$ replaced by $\boldsymbol{\Delta}^1_{2n+2}$, $\boldsymbol{\Sigma}^1_{2n+2}$ respectively.*

1.6. By examining the proof of Theorem 1.2.4 more carefully, we can actually extract a better estimate for the complexity of the uniformizing function G in the case $\sim \in \boldsymbol{\Delta}$, which is useful in certain applications. Let us establish some convenient notation first.

1.6.1. Definition. For each pointset A, there is a smallest Spector pointclass Γ such that $A \in \Delta$. It will be denoted by

$$\mathrm{ENV}(A),$$

(the *envelope* of A) as it coincides with the pointclass of sets Kleene semi-recursive in ${}^2E$, A.

1.6.2. Theorem. *Let Γ be a category-adequate Spector pointclass and $\sim$ a coarse equivalence relation on $W \subseteq \omega^\omega$. If $\sim \in \Delta$, there is $A \in \Delta$ such that for any $\sim$-invariant $R(w, x)$, with $\forall w \in W\, \exists x\, R(w, x)$, there is a uniformizing function $G : W \to \mathscr{X}$ which is $\boldsymbol{\Delta}^*$-measurable, where $\Gamma^* = \mathrm{ENV}(A)$. We are assuming here* $\mathrm{Det}(\Gamma^*; R)$.

Proof. Without loss of generality take $W = \omega^\omega$ (clearly $W \in \Delta$, as $\alpha \in W \Leftrightarrow \alpha \sim \alpha$). Let (viewing below σ ambiguously as a member of 2^ω and also as a strategy for player I) for $e \in \omega$ and $c \in 2^\omega$:

$$h(e, \sigma, c) = \sigma(\langle\langle e, \sigma\rangle, c\rangle).$$

Then put

$$B(e, \sigma, a) \Leftrightarrow \{b:\ h(e, \sigma, b) \sim h(e, \sigma, a)\} \text{ is not meager.}$$

Clearly $B \in \Delta$, as Γ is category-adequate. Moreover, as in the proof of Sublemma 1.4.3, since each $B_{e,\sigma} = \{a: B(e, \sigma, a)\}$ is not meager we can find $T(e, \sigma, s)$ in Δ such that for each e, σ, $T_{e,\sigma}$ is a perfect binary tree and $h_{e,\sigma}{}''[T_{e,\sigma}]$ falls in a single $\sim$-equivalence class.

Let now $A \in \Delta$ be a pointset encoding $\sim$, T. Put $\Gamma^* = \mathrm{ENV}(A)$. Then let $\mathscr{P}^*$ be the coding system for perfect binary trees associated with Γ^* as in the proof of Lemma 1.4.2. Then play the following game (recall that $W = \omega^\omega$ here):

I	II	II wins iff
w	$\langle a, c\rangle$	$\langle a, c\rangle$ is good (for $\mathscr{P}^*$)
		$\wedge R(w, a^*(c))$.

A winning strategy for II produces as before a uniformizing function which is $\boldsymbol{\Delta}^*$-measurable. To exclude the possibility that I has a winning strategy σ in this game, repeat the arguments in the proof of Lemma 1.4.2, working within Γ^* now and noticing that $T(e, \sigma, s)$ being in Δ^* is all that is needed.

1.7. Let us briefly comment, in concluding this section, on the concept of coarse equivalence relation. A conjecture is that all such definable equivalence relations are induced by norms, i.e. have the form $\sim\, = \,\sim_\phi$, for some $\phi : W \twoheadrightarrow \kappa$, granting determinacy hypotheses. This has been indeed verified for $\mathrm{IND}(\mathbb{R})$ and $\neg\mathrm{IND}(\mathbb{R})$ coarse equivalence relations on ω^ω by Harrington and Sami [3].

2. The Moschovakis Coding theorem revisited

2.1. Actually, the first general result on the question of uniformization of invariant pointsets $R(w, x)$, in the case $\sim = \sim_\phi$, was proved by Moschovakis [13]. He showed the following multivalued-uniformization theorem.

2.1.1. Theorem (Moschovakis [13]). *Let $\phi : W \twoheadrightarrow \kappa$ be a norm on $W \subseteq \omega^\omega$. Let $R(w, x)$ be $\sim_\phi$-invariant and assume $\forall w \in W \exists x R(w, x)$. Then granting* Det(HYP($\mathbb{R}$, R)), *there is $\bar{R}(w, x)$ also $\sim_\phi$-invariant such that*

$$\bar{R} \subseteq R \wedge \forall w \in W \exists x \bar{R}(w, x),$$

with $\bar{R} \in \boldsymbol{\Sigma}^1_1(\leqslant_\phi, \neg\leqslant_\phi)$, where $\boldsymbol{\Sigma}^1_1(A_1 \cdots A_n)$ is the smallest pointclass containing all the open pointsets, $A_1 \cdots A_n$ and closed under continuous substitutions $\vee$, $\wedge$, $\exists^\omega$, $\forall^\omega$ and $\exists^{\mathbb{R}}$.

Of course a uniformizing function G for R trivially produces a multivalued-uniformizing relation $\bar{R}$ as above by letting

$$\bar{R}(w, x) \Leftrightarrow \exists v (v \sim_\phi w \wedge x = G(v)).$$

Thus, *in certain cases,* Theorem 1.2.4 gives also as a corollary Theorem 2.1.1. For example, this happens when $\sim$ is $\boldsymbol{\Delta}^1_m$ or **HYP**($\mathbb{R}$). Although, in order to avoid a complicated statement, the determinacy hypotheses needed in Theorem 2.1.1 have been rather exaggerated, it is interesting to note that, when Theorem 1.2.4 is applicable, it requires a much lower dose of determinacy than what an accurate computation shows is needed for the original proof of Theorem 2.1.1.

A very important corollary that Moschovakis draws from Theorem 2.1.1 estimates the complexity of the coded versions of sets $P \subseteq \kappa$.

2.1.2. Corollary (Moschovakis [13]). *Let $\phi : W \twoheadrightarrow \kappa$ be a norm on $W \subseteq \omega^\omega$. Let $R(w_1 \cdots w_k)$ be $\sim_\phi$-invariant (i.e. $R(w_1 \cdots w_k) \wedge v_1 \sim_\phi w_1 \wedge \cdots \wedge v_k \sim_\phi w_k \Rightarrow R(v_1 \cdots v_k)$). Then, granting* Det(**HYP**($\mathbb{R}$, R)), *R is $\boldsymbol{\Delta}^1_1(\leqslant_\phi, \neg\leqslant_\phi)$.*

In this context and whenever again they are applicable, Theorems 1.2.4 and 1.6.2 give a better estimate in many instances than Corollary 2.1.2. The result below, proved by Harrington and Sami [3], has also interesting applications in the theory of equivalence relations.

2.1.3. Theorem (Harrington and Sami [3]). *Let $\sim$ be a course equivalence relation on $W \subseteq \omega^\omega$ and let $\sim \in \boldsymbol{\Gamma}$, where $\boldsymbol{\Gamma}$ is a category-adequate Spector pointclass. Then if $R(w_1 \cdots w_k)$ is $\sim$-invariant, we have that R is $\boldsymbol{\Delta}$ on W i.e. there are A, B in $\boldsymbol{\Gamma}$, $\neg\boldsymbol{\Gamma}$ respectively such that for $w_1 \cdots w_k \in W$, $R(w_1 \cdots w_k) \Leftrightarrow A(w_1 \cdots w_k) \Leftrightarrow B(w_1 \cdots w_k)$, granting* Det($\langle \boldsymbol{\Gamma}; \neg W, \neg R, R\rangle$).

If moreover $\sim \in \boldsymbol{\Delta}$, there is $A \in \boldsymbol{\Delta}$ such that $R \in \boldsymbol{\Delta}^*$, where $\boldsymbol{\Gamma}^* = \mathrm{ENV}(A)$.

Proof. Let us deal with the first case, as the second one is similar. Take $k=1$ for notational simplicity. Let

$$R'(w,\alpha)\Leftrightarrow(R(w)\wedge\alpha=\lambda t.0)\vee(\neg R(w)\wedge\alpha=\lambda t.1).$$

Let then, by Theorem 1.2.4 G be a uniformizing function for R', G $\boldsymbol{\Gamma}$-measurable. Then $R(w)\Leftrightarrow w\in W\wedge G(w)=\lambda t.0$. So $R\in\boldsymbol{\Gamma}$. Similarly, $\neg R\cap W\in\boldsymbol{\Gamma}$ and we are done.

2.2. Take for example the case $\sim=\sim_\phi$, where $\phi:W\twoheadrightarrow\kappa$ is a $\boldsymbol{\Delta}^1_m$-norm on the $\boldsymbol{\Delta}^1_m$ set W and $m\geq 2$. As for $A\in\boldsymbol{\Delta}^1_m$ we have $\mathrm{ENV}(A)\subseteq\boldsymbol{\Delta}^1_m$, every $\sim_\phi$-invariant $R(w_1\cdots w_k)$ belongs to $\boldsymbol{\Delta}^*$ for some $\Gamma^*\subseteq\boldsymbol{\Delta}^1_m$, i.e. all such R are not only $\boldsymbol{\Delta}^1_m$ as Corollary 2.1.2. asserts, but also of bounded complexity in the Wadge hierarchy of $\boldsymbol{\Delta}^1_m$ sets (i.e. there is fixed $S\in\boldsymbol{\Delta}^1_m$ so that all such R are reducible via continuous functions to S).

3. Ordinal quantification

3.1. One of the most interesting applications of Theorems 1.2.4 and 1.6.2 deals with the problem of invariant quantification. The main result is as follows:

3.1.1. Theorem. *Let $\sim$ be a coarse equivalence relation on $W\subseteq\omega^\omega$, Γ a category-adequate Spector pointclass. Assume moreover that $\sim\in\Delta$. If $\Gamma'\supseteq\Delta$ is any pointclass closed under recursive substitutions, $\wedge$, $\vee$, $\exists^\omega$, $\forall^\omega$ and $\forall^{\mathcal{R}}$ and if $R\in\exists^{\mathcal{R}}\Gamma'$ is $\sim$-invariant, then the following two pointsets are also in $\exists^{\mathcal{R}}\Gamma'$:*

$$\forall w\in W\,R(w,x),\quad \exists w\in W\,R(w,x),$$

granting $\mathrm{Det}(\Gamma')$.

Proof. That $\exists w\in W\,R(w,x)$ is in $\exists^{\mathcal{R}}\Gamma'$ is obvious. Let us prove now that also $\forall w\in W\,R(w,x)$ is in $\exists^{\mathcal{R}}\Gamma'$.

Let $R(w,x)\Leftrightarrow\exists\beta\,Q(w,x,\beta)$, where $Q\in\Gamma'$. First we will 'invariantize' Q. We have for $w\in W$,

$$\begin{aligned}R(w,x)&\Leftrightarrow\exists v\in W[R(v,x)\wedge v\sim w]\\&\Leftrightarrow\exists v\in W\,\exists\beta[Q(v,x,\beta)\wedge v\sim w]\\&\Leftrightarrow\exists\beta[Q((\beta)_0,s,(\beta)_1)\wedge(\beta)_0\sim w]\\&\Leftrightarrow\exists\beta\,P(w,x,\beta)\end{aligned}$$

where $P(w,x,\beta)\Leftrightarrow Q((\beta)_0,x,(\beta)_1)\wedge(\beta)_0\sim w$ is $\sim$-invariant (on w) and belongs to Γ'.

Now if $\forall w\in W\,R(w,x)$, we have $\forall w\in W\,\exists\beta\,P(w,x,\beta)$, so by Theorem 1.6.2 there is a Spector pointclass $\Gamma^*=\mathrm{ENV}(A)$, where $A\in\Delta$ and a $\boldsymbol{\Delta}^*$-measurable uniformizing function $G: W \to \omega^\omega$ for P. Then the relation

$$A_G(w, n, m) \Leftrightarrow w \in W \wedge G(w)(n) = m$$

is in $\boldsymbol{\Delta}^*$. So

$$\begin{aligned}&\forall w \in W \exists \beta\, P(w, x, \beta) \Leftrightarrow \exists A \subseteq \omega^\omega \times \omega \times \omega\\ &\{A \in \boldsymbol{\Delta}^* \wedge \forall w \in W \forall n\, \exists! m\, A(w, n, m)\\ &\quad \wedge \forall w \in W \forall \beta[\forall n\, \forall m(\beta(n) = m \Leftrightarrow A(w, n, m)) \Rightarrow P(w, x, \beta)]\}.\end{aligned}$$

To compute the complexity of this expression, let $U \subseteq (\omega^\omega)^2 \times \omega^2$ be in Γ^* and universal for the $\boldsymbol{\Gamma}^*$ subsets of $\omega^\omega \times \omega^2$. Put

$$U^0(\varepsilon, \alpha, m, n) \Leftrightarrow U((\varepsilon)_0, \alpha, m, n).$$

$$U^1(\varepsilon, \alpha, m, n) \Leftrightarrow U((\varepsilon)_1, \alpha, m, n)$$

and let $\bar{U}^0, \bar{U}^1 \in \Gamma^*$ reduce U^0, U^1. Then define

$$I(\varepsilon) \Leftrightarrow \bar{U}^0_\varepsilon \cup \bar{U}^1_\varepsilon = \omega^\omega \times \omega^2,$$

$$H(\varepsilon, \alpha, m, n) \Leftrightarrow \bar{U}^0(\varepsilon, \alpha, m, n),$$

$$\check{H}(\varepsilon, \alpha, m, n) \Leftrightarrow \neg \bar{U}^1(\varepsilon, \alpha, m, n).$$

Clearly, $I \in \forall^{\mathscr{R}} \Gamma^*$. But the closure properties of Γ' and the fact that $\Gamma^* = \mathrm{ENV}(A)$, where $A \in \Delta \subseteq \Delta'$ insure that $\Gamma^* \subseteq \Gamma'$, therefore $I \in \Gamma'$. Moreover, $H \in \Gamma^*$ and $\check{H} \in \neg\Gamma^*$. Finally, $\{I, H, \check{H}\}$ forms a coding system for subsets $A \subseteq \omega^\omega \times \omega^2$ in $\boldsymbol{\Delta}^*$ i.e.

(i) $I(\varepsilon) \Rightarrow H_\varepsilon = \check{H}_\varepsilon$.

(ii) $\{H_\varepsilon : \varepsilon \in I\} = \{A \subseteq \omega^\omega \times \omega^2 : A \in \boldsymbol{\Delta}^*\}$.

Thus we have

$$\begin{aligned}&\forall w \in W \exists \beta\, P(w, x, \beta) \Leftrightarrow \exists \varepsilon\{I(\varepsilon)\\ &\quad \wedge \forall w \in W \forall n\, \exists! m\, H(\varepsilon, w, n, m)\\ &\quad \wedge \forall w \in W \forall \beta[[\forall n\, \forall m(\beta(n) = m \Rightarrow \check{H}(\varepsilon, w, n, m))\\ &\quad \wedge \forall n\, \forall m(H(\varepsilon, w, n, m) \Rightarrow \beta(n) = m)] \Rightarrow P(w, x, \beta)]\},\end{aligned}$$

which is clearly in $\exists^{\mathscr{R}} \Gamma'$.

A more suggestive rephrasing of the conclusion of Theorem 3.1.1 is as follows: Given a coarse equivalence relation $\sim$ on $W \subseteq \omega^\omega$ and a pointclass Σ, say that a relation $P \subseteq K^n \times \mathscr{X}$ ($\mathscr{X}$ a product space and $K = K_\sim$) is *in* Σ, *relative to* $\sim$, if its coded version $P^*(w_1 \cdots w_n, x) \Leftrightarrow w_1 \cdots w_k \in W \wedge P(|w_1| \cdots |w_k|, x)$ is in Σ. Then Theorem 3.1.1 says (under the stated hypotheses) that $\exists^{\mathscr{R}} \Gamma'$ is closed under existential and universal quantification over K.

Of particular interest in the applications is the case $\sim = \sim_\phi$, where ϕ is a Δ^1_n-norm on a Δ^1_n set W and $\Gamma' = \Pi^1_n$.

3.1.2. Corollary. Assume $n \geq 1$ and $\mathrm{Det}(\boldsymbol{\Sigma}^1_n)$. Let $\phi : W \twoheadrightarrow \kappa$ be a Δ^1_n-norm on the Δ^1_n set W. Then the Σ^1_{n+1} relations are closed under existential and universal quantification over κ, relative to ϕ.

Proof. If n is odd, take in Theorem 3.1.1 $\Gamma = \Pi^1_n$, while if n is even take $\Gamma = \Sigma^1_n$. In both cases take $\Gamma' = \Pi^1_n$.

This result is in general best possible, as it is easy to construct a Δ^1_n-norm with range ω, such that Σ^1_n is not closed under universal quantification relative to it. Indeed, let $A \subseteq \omega - \{0\}$ be Σ^1_n but not Π^1_n and write $m \in A \Leftrightarrow \exists \alpha\, P(m, \alpha)$, where $P \in \Pi^1_{n-1}$. Consider now the following norm $\phi : \omega^\omega \to \omega$, where we put $\beta' = (\beta(1), \beta(2), \ldots)$:

$$\phi(\beta) = \begin{cases} \beta(0) & \text{if} \quad P(\beta(0), \beta'), \\ 0 & \text{if} \quad \neg P(\beta(0), \beta'). \end{cases}$$

Then if $R(\beta, m) \Leftrightarrow P(\beta(0), \beta') \wedge \beta(0) = m$, R is Π^1_{n-1} and it is $\sim_\phi$-invariant (on β). But since $m \in A \Leftrightarrow \exists \beta\, R(\beta, M)$, $\exists \beta\, R(\beta, m)$ is not Π^1_n, despite the fact that R is Π^1_{n-1}.

On the other hand for $n = 3$, it is known that for each Δ^1_3-norm $\phi : W \twoheadrightarrow \kappa$ on a Δ^1_3 set W, there is *another* Δ^1_3-norm $\phi^* : W \twoheadrightarrow \kappa$ such that $\boldsymbol{\Sigma}^1_3$ is closed under quantification on κ (of both types), relative to ϕ^* (see [7]). ($\mathrm{Det}(\boldsymbol{\Delta}^1_2)$ is needed for this result). If this holds for odd $n \geq 5$ is not yet known. It is easy to see, however, that such a property cannot hold for even n-otherwise Σ^1_n would be closed under wellordered intersections of length $\boldsymbol{\delta}^1_{n-1}$, granting AD (see Section 3.2).

3.2. It is easy now to combine Theorem 2.1.1 and Corollary 3.1.2, to derive from AD some boldface corollaries concerning closure of the projective pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$ under wellordered unions. Let us put down first the following corollary of Theorem 2.1.1.

3.2.1. Lemma (Moschovakis [13]). *Assume* AD. *Let $n \geq 1$, $\kappa < \delta^1_n$ and let $\{A_\xi\}_{\xi<\kappa}$ be a sequence of $\boldsymbol{\Sigma}^1_n(\boldsymbol{\Pi}^1_n, \boldsymbol{\Delta}^1_n)$ sets. Fix $\phi : W \twoheadrightarrow \kappa$ a $\boldsymbol{\Delta}^1_n$-norm on $W \in \boldsymbol{\Delta}^1_n$. Then the relation*

$$A(w, \alpha) \Leftrightarrow w \in W \wedge \alpha \in A_{\phi(w)}$$

is also $\boldsymbol{\Sigma}^1_n(\boldsymbol{\Pi}^1_n, \boldsymbol{\Delta}^1_n)$.

Proof. Let $U \subseteq \omega^\omega \times \omega^\omega$ be universal $\boldsymbol{\Sigma}^1_n$. Put

$$R(w, \varepsilon) \Leftrightarrow w \in W \wedge U_\varepsilon = A_{\phi(w)}.$$

Clearly $\forall w \in W\, \exists \varepsilon\, R(w, \varepsilon)$. Then by Theorem 2.1.1 find $\bar{R}(w, \varepsilon)$ in $\boldsymbol{\Sigma}^1_n$ which is

$\sim_\phi$-invariant, $\tilde{R} \subseteq R$ and $\forall w \in W \exists \varepsilon \tilde{R}(w, \varepsilon)$. Then

$$A(w, \alpha) \Leftrightarrow \exists \varepsilon[\tilde{R}(w, \varepsilon) \wedge \alpha \in U_\varepsilon],$$

so $A \in \boldsymbol{\Sigma}^1_n$.

For the $\boldsymbol{\Pi}^1_n$ case, put

$$R(w, \varepsilon) \Leftrightarrow w \in W \wedge \neg U_\varepsilon = A_{\phi(w)}$$

and use instead the formula

$$A(w, \alpha) \Leftrightarrow \forall \varepsilon[\tilde{R}(w, \varepsilon) \Rightarrow \alpha \notin U_\varepsilon].$$

Finally, for the $\boldsymbol{\Delta}^1_n$ case let

$$R(w, \varepsilon) \Leftrightarrow w \in W \wedge U_{(\varepsilon)_0} = \neg U_{(\varepsilon)_1} \wedge U_{(\varepsilon)_0} = A_{\phi(w)}$$

and use both the above formulas.

We have now

2.2. Corollary. *Assume AD and $n \geq 1$. Then if $\kappa < \boldsymbol{\delta}^1_n$ and $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\boldsymbol{\Pi}^1_{n+1}$ sets, $\bigcup_{\xi<\kappa} A_\xi$ is also $\boldsymbol{\Pi}^1_{n+1}$.*

Proof. Let $W \in \boldsymbol{\Delta}^1_n$, $\phi : W \twoheadrightarrow \kappa$ be a $\boldsymbol{\Delta}^1_n$-norm. By Lemma 3.2.1 we have that $A(w, \alpha) \Leftrightarrow w \in W \wedge \alpha \in A_{\phi(w)}$ is also $\boldsymbol{\Pi}^1_{n+1}$. Then $\alpha \in \bigcup_{\xi<\kappa} A_\xi \Leftrightarrow \exists w \in W A(w, \alpha)$, so by (the relativized Corollary 3.1.2) $\bigcup_{\xi<\kappa} A_\xi$ is $\boldsymbol{\Pi}^1_{n+1}$.

The following related results are also known, granting always AD:

(i) For $n \geq 2$ even, if $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\boldsymbol{\Sigma}^1_n$ sets of *arbitrary* length, $\bigcup_{\xi<\kappa} A_\xi$ is also $\boldsymbol{\Sigma}^1_n$ (see [6]).

(ii) For n odd, if $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\kappa < \boldsymbol{\delta}^1_n$ $\boldsymbol{\Sigma}^1_n$ sets, $\bigcup_{\xi<\kappa} A_\xi$ is also $\boldsymbol{\Sigma}^1_n$, but every $\boldsymbol{\Sigma}^1_{n+1}$ set is the union of a sequence $\{A_\xi\}_{\xi<\boldsymbol{\delta}^1_n}$ of $\boldsymbol{\Delta}^1_n$ sets (see [13]).

(iii) For $n = 3$, if $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\kappa < \boldsymbol{\delta}^1_3$ $\boldsymbol{\Pi}^1_3$ sets, then $\bigcup_{\xi<\kappa} A_\xi$ is $\boldsymbol{\Pi}^1_3$ (see [7]). But it is not known if for odd $n \geq 5$, if $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\kappa < \boldsymbol{\delta}^1_n$ $\boldsymbol{\Pi}^1_n$ sets, then $\bigcup_{\xi<\kappa} A_\xi$ is also $\boldsymbol{\Pi}^1_n$.

(iv) If n is odd and $\{A_\xi\}_{\xi<\kappa}$ is a sequence of $\kappa < \boldsymbol{\delta}^1_n$ $\boldsymbol{\Delta}^1_n$ sets, then $\bigcup_{\xi<\kappa} A_\xi$ is also $\boldsymbol{\Delta}^1_n$ (see [11]).

3.3. We can also use Corollary 3.1.2 to derive a rank comparison theorem for projective prewellorderings and wellfounded relations (weaker forms of these results can be deduced also from Theorem 2.1.1 using much stronger determinacy hypotheses).

3.3.1. Lemma. *Assume $n \geq 1$ and* Det($\boldsymbol{\Sigma}^1_n$). *If $\prec$ is a Σ^1_{n+1} well-founded relation on ω^ω and $\phi : W \twoheadrightarrow \kappa$ a Δ^1_n-norm on the Δ^1_n set $W \subseteq \omega^\omega$, the relation*

$$y \in W \wedge [x \notin \mathrm{Field}(\prec) \vee \mathrm{rank}_\prec(x) \leq \phi(y)]$$

is Π^1_{n+1}.

Proof. Consider the following monotone operator $\Psi(x, S)$, where $x \in \omega^\omega$, $S \subseteq \omega^\omega$:

$$\Psi(x, S) \Leftrightarrow \forall x'(x' < x \Rightarrow x' \in S].$$

Let Ψ^ξ be the ξth iterate of Ψ. Then $\Psi^\xi(x) \Leftrightarrow x \notin \text{Field}(<) \vee \text{rank}_<(x) \leq \xi$. We shall prove now that the relation

$$R(x, y) \Leftrightarrow y \in W \wedge x \in \Psi^{\phi(y)}$$

is Π^1_{n+1}, which completes the proof.

To compute the complexity of R we use the Recursion theorem. Let $U(e, x, y)$ be universal for the Π^1_{n+1} subsets of $(\omega^\omega)^2$. We need to find $e^* \in \omega$ with $R = U_{e^*}$. But note that

$$\begin{aligned} R(x, y) &\Leftrightarrow y \in W \wedge x \in \Psi^{\phi(y)} \\ &\Leftrightarrow y \in W \wedge \Psi(x, \textstyle\bigcup_{\xi < \phi(y)} \Psi^\xi) \\ &\Leftrightarrow y \in W \wedge \forall x'[x' < x \Rightarrow \exists y'(\phi(y') < \phi(y) \wedge x' \in \Psi^{\phi(y')})] \\ &\Leftrightarrow y \in W \wedge \forall x'(x' < x \Rightarrow \exists y'(\phi(y') < \phi(y) \wedge R(x', y'))]. \end{aligned}$$

So we need e^* satisfying

$$\begin{aligned} U_{e^*}(x, y) &\Leftrightarrow y \in W \wedge \forall x'[x' < x \Rightarrow \exists y'(\phi(y') < \phi(y) \\ &\quad \wedge \forall y''[\phi(y'') = \phi(y') \Rightarrow U(e^*, x', y'')]] \\ &\Leftrightarrow T(e^*, x, y), \end{aligned}$$

where $T(e, x, y)$ is Π^1_{n+1} by Corollary 3.1.2. (The use of $\forall y''[\cdots]$ makes sure that $\exists y'$ is applied to an invariant matrix.) The Recursion Theorem guarantees the existence of such an e^* and then an easy induction on $\text{rank}_<(x)$ shows that $U_{e^*}(x, y) \Leftrightarrow R(x, y)$ and the proof is complete.

We have therefore

3.3.2. Theorem. *Assume $n \geq 1$ and $\text{Det}(\boldsymbol{\Sigma}^1_n)$.*

(i) *If $\phi : W \twoheadrightarrow \kappa$, $\phi' : W' \twoheadrightarrow \kappa'$ are Δ^1_n-norms on the Δ^1_n sets W, W', then the relation*

$$w \in W \wedge w' \in W' \wedge \phi(w) \leq \phi'(w')$$

is Δ^1_{n+1}.

(ii) *If $\text{Det}(\boldsymbol{\Sigma}^1_{n+1})$ holds and $<$, $<'$ are wellfounded Σ^1_n relations, then the predicate*

$$x \in \text{Field}(<) \wedge x' \in \text{Field}(<') \wedge \text{rank}_<(x) \leq \text{rank}_{<'}(x')$$

is Δ^1_{n+2}.

Proof. (i) By Lemma 3.3.1, the relation

$$w \in W \wedge w' \in W' \wedge \phi(w) \leq \phi'(w')$$

is Π^1_{n+1}. But since

$$w' \in W' \wedge w \in W \wedge \phi'(w') < \phi(w)$$
$$\Leftrightarrow w' \in W' \wedge w \in W \wedge \exists v \in W(\phi(v) < \phi(w) \wedge \phi'(w') \leq \phi(v)),$$

this strict relation is also Π^1_{n+1} by Corollary 3.1.2 and we are done.

(ii) Let $\phi : \omega^\omega \twoheadrightarrow \mu$ be a Δ^1_n-norm such that $\max\{\mathrm{rank}(<), \mathrm{rank}(<')\} < \mu$. Note that

$$x \in \mathrm{Field}(<) \wedge \mathrm{rank}_<(x) = \phi(y) \Leftrightarrow x \in \mathrm{Field}(<) \wedge \mathrm{rank}_<(x) \leq \phi(y)$$
$$\wedge \neg\exists z[\phi(z) < \phi(y) \wedge \mathrm{rank}_<(x) \leq \phi(z)],$$

so that this relation is Δ^1_{n+2}. Then

$$x \in \mathrm{Field}(<) \wedge x' \in \mathrm{Field}(<') \wedge \mathrm{rank}_<(x) \leq \mathrm{rank}_{<'}(x')$$
$$\Leftrightarrow x \in \mathrm{Field}(<) \wedge x' \in \mathrm{Field}(<')$$
$$\wedge \exists y\, \exists y'[\mathrm{rank}_<(x) = \phi(y) \wedge \mathrm{rank}_{<'}(x') = \phi(y') \wedge \phi(y) \leq \phi(y')],$$

and therefore we are done using Corollary, 3.1.2 again.

The following invariance result is an immediate corollary of Theorem 3.3.2.

3.3.3. Corollary. *Assume* $n \geq 1$ *and* $\mathrm{Det}(\boldsymbol{\Sigma}^1_n)$. *Let* $\phi : W \twoheadrightarrow \kappa$, $\phi' : W' \twoheadrightarrow \kappa$ *be* Δ^1_n*-norms on the* Δ^1_n *sets* W, W' *with the same range* κ. *If* $P(\xi_1 \cdots \xi_n, x)$ *is a relation, where* $\xi_i < \kappa$ *and* $x \in \mathscr{X}$, *then* P *is* Σ^1_{n+1} *relative to* $\sim_\phi$ *iff it is* Σ^1_{n+1} *relative to* $\sim_{\phi'}$.

Proof. Let P^*_ϕ, $P^*_{\phi'}$ be the two coded versions of P. Then

$$P^*_\phi(w_1 \cdots w_k, x) \Leftrightarrow w_1 \cdots w_k \in W$$
$$\wedge \exists w'_1 \cdots w'_k \in W' \left[\bigwedge_{i=1}^{k} \phi(w_i) = \phi'(w'_i) \wedge P^*_{\phi'}(w'_1 \cdots w'_k)\right].$$

3.4. A quite different invariance theorem is also a consequence of Corollary 3.1.2. Some notation first.

3.4.1. Definition. For each odd n, let

$$\lambda_n(\alpha) = \sup\{\xi : \xi \text{ is the rank of a } \Delta^1_n(\alpha) \text{ prewellordering of } \omega^\omega\}.$$

Then for n even, put

$$\mu_n(\alpha) = \sup\{\lambda_{n-1}(\beta) : \beta \in \Delta^1_n(\alpha)\}.$$

Clearly, $\lambda_1(\alpha) = \omega_1(\alpha) =$ first non-recursive in α ordinal and $\mu_2(\alpha) = \delta^1_2(\alpha) = \sup\{\xi : \xi$ is the rank of a $\Delta^1_2(\alpha)$ wellordering of $\omega\}$ = first stable in α ordinal. Thus λ_n, μ_n are higher level analogs of these basic notions of effective descriptive set theory.

A well-known theorem of Moschovakis establishes the following invariance property of Π^1_{2n+1}-norms.

3.4.2. Theorem (Moschovakis). *Assume* Det($\boldsymbol{\Delta}^1_{2n}$). *Then if* $\mathcal{P} \subseteq \omega^\omega$ *is a complete* Π^1_{2n+1} *set and* $\phi : \mathcal{P} \twoheadrightarrow \boldsymbol{\delta}^1_{2n+1}$ *a* Π^1_{2n+1}*-norm on* $\mathcal{P}$, *we have for each* α,

$$\sup\{\phi(x)\colon x \in \mathcal{P} \wedge x \leqslant_T \alpha\} = \sup\{\phi(x)\colon x \in \mathcal{P} \wedge x \in \Delta^1_{2n+1}(\alpha)\} = \lambda_{2n+1}(\alpha).$$

We will prove below a somewhat analogous result for Σ^1_{2n}-norms and the assignments $\mu_{2n}(\alpha)$. We need some terminology first.

3.4.3. Definition. Let $\mathcal{S}$ be a Σ^1_{2n+2} subset of ω^ω. A Σ^1_{2n+2}-norm $\phi : \mathcal{S} \twoheadrightarrow \kappa$ on $\mathcal{S}$ is called *induced* if there is a Π^1_{2n+1} set $\mathcal{P}$ and a Π^1_{2n+1}-norm $\psi : \mathcal{P} \twoheadrightarrow \lambda$ on $\mathcal{P}$ such that

(i) $x \in \mathcal{S} \Leftrightarrow \exists \beta\, \mathcal{P}(x, \beta)$,

(ii) if $\phi^*(x) = \min\{\psi(x, \beta)\colon \mathcal{P}(x, \beta)\}$, then $\leqslant_\phi = \leqslant_{\phi^*}$. (Note that ϕ^* need not have range an initial segment of ordinals.)

We have now

3.4.4. Theorem. *Assume* Det($\boldsymbol{\Delta}^1_{2n+3}$). *Let* $\mathcal{S} \subseteq \omega^\omega$ *be a* Σ^1_{2n+2} *but not* $\boldsymbol{\Delta}^1_{2n+2}$ *set and let* $\phi : \mathcal{S} \twoheadrightarrow \kappa$ *be a* Σ^1_{2n+2}*-norm on* $\mathcal{S}$. *Then*

(i) $\sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \leqslant_T \alpha\} = \sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\} \geqslant \mu_{2n+2}(\alpha)$.

(ii) *If* ϕ *is induced, then*

$$\sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\} = \mu_{2n+2}(\alpha).$$

Proof. It is easy to check by standard prewellordering arguments, that $\sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \leqslant_T \alpha\} = \sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\}$. Moreover, if ϕ is induced, then for $x \in \mathcal{S}$ and $x \in \Delta^1_{2n+2}(\alpha)$, we have $\phi(x) \leqslant \phi^*(x) = \min\{\psi(x, \beta)\colon \mathcal{P}(x, \beta)\}$, where $\mathcal{P}$, ψ are as above. But for such a x, there is $\beta \in \Delta^1_{2n+2}(x)$ with $\mathcal{P}(x, \beta)$, by the basis theorem. Then clearly, $\psi(x, \beta) < \lambda_{2n+1}(x, \beta) < \mu_{2n+2}(x)$ and so $\phi(x) < \mu_{2n+2}(x)$. This shows that $\sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\} \leqslant \mu_{2n+2}(\alpha)$.

So it only remains to prove that

$$\sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\} \geqslant \mu_{2n+2}(\alpha).$$

Assume not, towards a contradiction. Then we can find $\alpha_0 \in \Delta^1_{2n+2}(\alpha)$ and $\chi : \omega^\omega \twoheadrightarrow \rho$, where χ is a $\Delta^1_{2n+1}(\alpha_0)$-norm and $\rho > \sup\{\phi(x)\colon x \in \mathcal{S} \wedge x \in \Delta^1_{2n+2}(\alpha)\}$. By Lemma 3.3.1 the relation

$$x \notin \mathcal{S} \vee \phi(x) \leqslant \chi(y)$$

is $\Pi^1_{2n+2}(\alpha_0)$, so

$$x \notin \mathcal{S} \vee \phi(x) < \rho \Leftrightarrow R(x)$$

is also $\Pi^1_{2n+2}(\alpha_0)$. If $\mathscr{S}-R\neq\emptyset$, then as it is a $\Sigma^1_{2n+2}(\alpha_0)$ set, it contains a real $x\in\Delta^1_{2n+2}(\alpha)$. Then $\phi(x)\geqslant\rho$, a contradiction. So $\mathscr{S}\subseteq R$, i.e. $x\in\mathscr{S}\Rightarrow\phi(x)<\rho$.

Thus, there is a $\mathbf{\Delta}^1_{2n+1}$-norm $\psi:\omega^\omega\twoheadrightarrow\kappa$, since $\kappa\leqslant\rho<\delta^1_{2n+1}$. Put

$$P(w,x)\Leftrightarrow x\in\mathscr{S}\wedge\phi(x)=\psi(w).$$

Then P is $\mathbf{\Delta}^1_{2n+3}$ by Theorem 3.3.2. Moreover, $\forall w\,\exists x\,P(w,x)$ and P is $\sim_\psi$-invariant on w. Thus by Corollary 1.5.1, there is a $\mathbf{\Delta}^1_{2n+1}$ function $G:\omega^\omega\to\omega^\omega$ such that $\forall w(G(w)\in\mathscr{S}\wedge\phi(G(w))=\psi(w))$. Thus

$$x\in\mathscr{S}\Leftrightarrow\exists w\,T(x,G(w)),$$

where T is a Π^1_{2n+2} relation such that for $y\in\mathscr{S}$,

$$T(x,y)\Leftrightarrow x\in\mathscr{S}\wedge\phi(x)=\phi(y).$$

Now note that if $w\sim_\psi w'$, then $G(w)\sim_\phi G(w')$, so $T(x,G(w))\Leftrightarrow T(x,G(w'))$, i.e. the relation $Q(w,x)\Leftrightarrow T(x,G(w))$ is $\sim_\psi$-invariant and as a consequence of Corollary 3.1.2 $\mathscr{S}\in\mathbf{\Pi}^1_{2n+2}$, a contradiction.

4. The computation of the reals in $L[T^{2n+1}]$

4.1. Let $\mathscr{P}^{2n+1}$ be a complete Π^1_{2n+1} set of reals and, assuming $\mathrm{Det}(\mathbf{\Delta}^1_{2n})$, let $\{\phi_m\}$ be a Π^1_{2n+1}-scale on $\mathscr{P}^{2n+1}$, $\phi_m:\mathscr{P}^{2n+1}\twoheadrightarrow\boldsymbol{\delta}^1_{2n+1}$. Let T^{2n+1} be its associated tree (on $\omega\times\boldsymbol{\delta}^1_{2n+1}$):

$$T^{2n+1}=\{(\alpha(0),\phi_0(\alpha),\ldots,\alpha(m-1),\phi_{m-1}(\alpha)):\ m\in\omega,\ \alpha\in\mathscr{P}^{2n+1}\}.$$

As it turns out $L[T^1]=L$ (*see* [8]) and in many ways $L[T^{2n+1}]$ serves as an analog of L for the level $2n+2$ of the analytical hierarchy (for example every Σ^1_{2n+2} set in Souslin over $L[T^{2n+1}]$). It is therefore of interest to be able to identify $\omega^\omega\cap L(T^{2n+1}]$, the set of reals in $L[T^{2n+1}]$. As $\omega^\omega\cap L[T^1]=\omega^\omega\cap L=C_2=$ the largest countable Σ^1_2 set, it is natural to wonder, following periodicity, whether or not in general $L[T^{2n+1}]\cap\omega^\omega=C_{2n+2}=$ the largest countable Σ^1_{2n+2} set. An affirmative answer has been conjectured by Moschovakis and our goal in this section is to prove this conjecture.

4.1.1. Theorem. *Assume* $\mathrm{Det}(P(\omega^\omega)\cap L[\omega^\omega])$. *Then* $\omega^\omega\cap L(T^{2n+1}]=C_{2n+2}$.

Some comments are in order before we give the proof:

(i) The case $n=1$ was proved in [7], before the general case.

(ii) In this section we shall give a proof of this result using $\mathrm{Det}(P(\omega^\omega)\cap L[\omega^\omega])$. This proof is based on Theorem 2.1.1 (whose use in this context was also initiated by Moschovakis) and the ordinal quantification results of the preceding section. It has the advantage of being conceptually simple (despite some lengthy computations), resembling in several points the usual computation that $\omega^\omega\cap L$ is Σ^1_2. On the other hand the result can be proved in PD only. Actually Det($\boldsymbol{\Delta}^1_{2n+2}$) is sufficient for the computation of $\omega^\omega \cap L[T^{2n+1}]$ and this proof, which is certainly more complicated, is given in Part II, as it uses also some ideas and results from the theory of ordinal games presented there.

4.2. Proof. As the theorem we want to prove is absolute for $L[\omega^\omega]$, we can work entirely *within* $L[\omega^\omega]$. By our assumption ZF+DC+AD holds in this model and so from now on we shall actually work entirely in this theory.

Since it is well-known, by Mansfield's theorem, that $C_{2n+2} \subseteq L[T^{2n+1}]$, it is enough to prove that $\omega^\omega \cap L[T^{2n+1}] \subseteq C_{2n+2}$. Since $\omega^\omega \cap L[T^{2n+1}]$ carries a wellordering, AD implies that it is countable, so it is enough to compute that

$$(*) \qquad \omega^\omega \cap L[T^{2n+1}] \text{ is } \Sigma^1_{2n+2}.$$

4.3. Let us first recall some standard terminology. Given a set A, a *second-order relation* on A is a relation $\mathcal{P}(\bar{x}, \bar{R})$, where $\bar{x} = (x_1 \cdots x_n)$ varies over some A^n and $\bar{R} = (R_1 \cdots R_k)$ over some product $P(A^{n_1}) \times \cdots P(A^{n_k})$, i.e. R_i is a n_i-ary relation to A. Put also $\mathcal{WF}(R) \Leftrightarrow R \subseteq A^2$ is well founded and $\mathcal{WO}(R) \Leftrightarrow R \subseteq A^2$ is a wellordering.

The plan of the proof of $(*)$ is now as follows:

(A) We shall find first an ordinal $\kappa < \boldsymbol{\delta}^1_{2n+1}$ for which $\kappa^+ = \boldsymbol{\delta}^1_{2n+1}$ and there is $\phi : \omega^\omega \twoheadrightarrow \kappa$ a Δ^1_{2n+1}-norm with range κ. Using Corollary 2.1.2, we shall define for each m a nice coding $R^m : \mathrm{CD}^m \twoheadrightarrow P(\kappa^m)$ of all the m-ary relations on κ by reals (i.e. $\mathrm{CD}^m \subseteq \omega^\omega$). Put

$$\Gamma = \{\mathcal{P}(\bar{\xi}, \bar{R}) : \mathcal{P} \text{ is a second order relation on } \kappa, \text{ which is } \Sigma^1_{2n+2} \text{ in the codes (provided by } \phi \text{ and } \{R^m\})\}.$$

Using Section 3, we shall prove that Γ contains the elementary second order relations on the structure $\langle \kappa, < \rangle$ $\mathcal{WF}$, $\neg\mathcal{WF}$ and is closed under $\wedge$, $\vee$, $\exists\xi$, $\forall\xi$ and existential quantification over relations on κ.

(B) For $R \in \mathcal{WO}$, let $|R| = \mathrm{rank}(R)$. This introduces codes of the ordinals $<\boldsymbol{\delta}^1_{2n+1}$, via relations on κ. We shall prove that T^{2n+1} (viewed as a subset of $\boldsymbol{\delta}^1_{2n+1}$) is Δ in these codes.

(C) Finally, by standard arguments, we shall show that if $A \subseteq \boldsymbol{\delta}^1_{2n+1}$ is Δ in the codes, then the second-order relation,

$$\mathcal{L}(X) \Leftrightarrow X \subseteq \kappa \wedge X \in L[A]$$

is in Γ. This together with (A) completes the proof.

4.4. We start with a proof of (A).

4.4.1. Lemma. *There is a Δ^1_{2n+1}-norm $\phi : \omega^\omega \twoheadrightarrow \kappa$, where $\kappa < \boldsymbol{\delta}^1_{2n+1}$ and $\kappa^+ = \boldsymbol{\delta}^1_{2n+1}$.*

Proof. Let $\{\psi_m\}$ be a Δ^1_{2n+1}-scale on a complete Π^1_{2n} set Q, $\psi_m : Q \twoheadrightarrow \lambda_m$. Let $\kappa = \sup_m \lambda_m$. Then κ is the rank of a Δ^1_{2n+1} prewellordering of ω^ω, so $\kappa < \boldsymbol{\delta}^1_{2n+1}$. Let $\phi : \omega^\omega \twoheadrightarrow \kappa$ be a Δ^1_{2n+1}-norm. Since every Π^1_{2n} set is κ-Souslin, every $\boldsymbol{\Sigma}^1_{2n+1}$ set is μ-Souslin for some $\mu < \kappa^+$. Then, by Kunen–Martin, every $\boldsymbol{\Sigma}^1_{2n+1}$ wellfounded relation has rank $< \kappa^+$, so $\kappa^+ = \boldsymbol{\delta}^1_{2n+1}$.

Let us fix κ and ϕ from now on and put as usual $|w| = \phi(w)$.

We will define next the set CD^m of codes for m-ary relations on κ, using Corollary 2.1.2. First let $U(\varepsilon, \bar\alpha)$ be Π^1_{2n+1} and universal for the Π^1_{2n+1} subsets of $(\omega^\omega)^m$. Let $U^0(\varepsilon, \bar\alpha) \Leftrightarrow U((\varepsilon)_0, \bar\alpha)$, $U^1(\varepsilon, \bar\alpha) \Leftrightarrow U((\varepsilon)_1, \bar\alpha)$ and reduce U^0, U^1 to $\bar U^0$, $\bar U^1$ also in Π^1_{2n+1}. Call ε a $\boldsymbol{\Delta}^1_{2n+1}$-*code*^{m} if $\bar U^0_\varepsilon \cup \bar U^1_\varepsilon = (\omega^\omega)^m$ and let $H^m_\varepsilon \equiv H_\varepsilon = \bar U^0_\varepsilon (= \neg \bar U^1_\varepsilon)$ be the $\boldsymbol{\Delta}^1_{2n+1}$ set coded by ε. The set C^m of $\boldsymbol{\Delta}^1_{2n+1}$-codes^m is clearly Π^1_{2n+1}. Put now

$$\varepsilon \in \mathrm{CD}^m \Leftrightarrow \varepsilon \text{ is a } \boldsymbol{\Delta}^1_{2n+1}\text{-code}^m \wedge \forall w_1 \cdots w_m\, \forall w'_1 \cdots w'_m \Big[\bigwedge_{i=1}^m |w_i| = |w'_i| \Rightarrow ((w_1 \cdots w_m) \in H^m_\varepsilon \Leftrightarrow (w'_1 \cdots w'_m) \in H^m_\varepsilon) \Big].$$

Clearly $\mathrm{CD}^m \in \Pi^1_{2n+1}$. For $\varepsilon \in \mathrm{CD}^m$, put now

$$R^m(\varepsilon) \equiv R_\varepsilon = \{(|w_1| \cdots |w_m|) : (w_1 \cdots w_m) \in H_\varepsilon\}$$

(dropping obvious embellishments). By Corollary 2.1.2

$$R^m : \mathrm{CD}^m \twoheadrightarrow P(\kappa^m).$$

Finally define the following class Γ of second order relations on κ:

For each $\mathscr{P}(\bar\xi, \bar R)$, let $\mathscr{P}^*(\bar w, \bar\varepsilon) \Leftrightarrow \varepsilon_1 \in \mathrm{CD}^{n_1} \wedge \cdots \wedge \varepsilon_k \in \mathrm{CD}^{n_k} \wedge \mathscr{P}(|w_1| \cdots |w_n|, R_{\varepsilon_1}, \ldots, R_{\varepsilon_k})$. Then put

$$\Gamma = \{\mathscr{P}(\bar\xi, \bar R) : \mathscr{P}^* \text{ is } \Sigma^1_{2n+2}\}.$$

We conclude part (A) by proving the following

4.4.2 Lemma. *Γ contains all the elementary second-order relations on $\langle \kappa, > \rangle$, $\mathscr{WF}$, $\neg\mathscr{WF}$ and is closed under $\wedge$, $\vee$, $\exists\xi$, $\forall\xi$ and existential quantification over relations on κ.*

Proof. To see that $\mathscr{WF} \in \Gamma$ we compute:

$$\begin{aligned} \mathscr{WF}^* &\Leftrightarrow \varepsilon \in \mathrm{CD}^2 \wedge R_\varepsilon \text{ is wellfounded} \\ &\Leftrightarrow \varepsilon \in \mathrm{CD}^2 \wedge \neg\exists\xi_0, \exists\xi_1, \xi_2, \ldots, \forall n\, R_\varepsilon(\xi_{n+1}, \xi_n) \\ &\Leftrightarrow \varepsilon \in \mathrm{CD}^2 \wedge \neg\exists\alpha\, \forall n ((\alpha)_{n+1}, (\alpha)_n) \in H_\varepsilon, \end{aligned}$$

which is clearly in Π^1_{2n+1}.

For closure under $\forall\xi$, let $\mathscr{P}(\bar\eta, \bar R, \xi)$ be in Γ. We have to prove that $\mathscr{L}(\bar\eta, \bar R) \Leftrightarrow$

$\forall \xi\, \mathscr{P}(\bar{\eta}, \bar{R}, \xi)$ is also in Γ. Indeed,

$$\mathscr{L}^*(\bar{w}, \bar{\varepsilon}) \Leftrightarrow \forall v\, \mathscr{P}^*(\bar{w}, \bar{\varepsilon}, v)$$

and as $P^* \in \Sigma^1_{2n+2}$ and is $\sim_\phi$-invariant on v, we have that $\mathscr{L}^* \in \Sigma^1_{2n+2}$, by Corollary 3.1.2.

The other assertions of this lemma are obvious.

4.5. We come now to part (B). For $R \in \mathscr{WO}$, let $|R| = \text{rank}(R)$. Thus $\{|R|: R \in \mathscr{WO}\} = \boldsymbol{\delta}^1_{2n+1}$ and we have a coding system for ordinals $<\boldsymbol{\delta}^1_{2n+1}$ with set of codes $\mathscr{WO} \in \Delta$. For $A \subseteq \boldsymbol{\delta}^1_{2n+1}$, let

$$\tilde{A} = \{R \in \mathscr{WO}: |R| \in A\},$$

be the coded version of A. We want to compute now the complexity of T^{2n+1} in this coding. First we shall code T^{2n+1}, which is literally a subset of $(\omega \times \boldsymbol{\delta}^1_{2n+1})^{<\omega}$, as a subset of $\boldsymbol{\delta}^1_{2n+1}$. For that fix some canonical bijection

$$\langle\,\rangle: (\omega \times \boldsymbol{\delta}^1_{2n+1})^{<\omega} \rightarrowtail\!\!\!\!\rightarrow \boldsymbol{\delta}^1_{2n+1}$$

and put

$$T = \{\langle k_0, \xi_0, \ldots, k_{m-1}, \xi_{m-1}\rangle: (k_0, \xi_0, \ldots, k_{m-1}, \xi_{m-1}) \in T^{2n+1}\}.$$

As $L[T] = L[T^{2n+1}]$, we shall work with T from now on. Notice that it is easy to arrange things (any reasonable choice of $\langle\,\rangle$ will do) so that $\langle\,\rangle$ is Δ in the codes in the following sense.

4.5.1. Lemma. *The following relation is in Δ (where for $S \subseteq \kappa^3$, $S_i = \{(\xi, \eta): S(i, \xi, \eta)\}$):*

$$U(m, s, R, S) \Leftrightarrow m \in \omega \wedge s \in \omega \wedge R \in \mathscr{WO} \wedge S \subseteq \kappa^3$$
$$\wedge \forall i < m (S_i \in \mathscr{WO}) \wedge |R| = \langle (s)_0, |S_0|, \ldots, (s)_{m-1}, |S_{m-1}|\rangle.$$

We can now compute the complexity of T:

4.5.2. Lemma. $\tilde{T} \in \Delta$.

Proof. We have to show that $\mathscr{T} = (\tilde{T})^*$ is Δ^1_{2n+2}. By definition,

$$\begin{aligned}\varepsilon \in \mathscr{T} \Leftrightarrow{}& \varepsilon \in \text{CD}^2 \wedge R_\varepsilon \in \mathscr{WO} \wedge \exists \alpha \{\alpha \in \mathscr{P}^{2n+1} \wedge \exists m\, \exists \delta [\delta \in \text{CD}^3 \\ & \wedge \forall i < m ((R_\delta)_i \in \mathscr{WO}) \wedge |R_\varepsilon| \\ & \qquad = \langle \alpha(0), |(R_\delta)_0|, \ldots, \alpha(m-1), (R_\delta)_{m-1}\rangle \\ & \wedge \forall i < m (\phi_i(\alpha) = |(R_\delta)_i|)]\} \\ \Leftrightarrow{}& \varepsilon \in \mathscr{WO}^* \wedge \exists \alpha \{\alpha \in \mathscr{P}^{2n+1} \wedge \exists m\, \exists \delta [U^*(m, \bar{\alpha}(m), \varepsilon, \delta) \\ & \wedge \forall i < m (\phi_i(\alpha) = |(R_\delta)_i|)]\},\end{aligned}$$

so that $\mathcal{T}$ is Σ^1_{2n+2}, granting that

$$V(i, \varepsilon, \alpha) \Leftrightarrow \varepsilon \in \mathcal{WO}^* \wedge \alpha \in \mathcal{P}^{2n+1} \wedge \phi_i(\alpha) = |R_\varepsilon|$$

is Σ^1_{2n+2}, which we shall indeed check below. But since $\phi_i : \mathcal{P}^{2n+1} \twoheadrightarrow \boldsymbol{\delta}^1_{2n+1}$ has range *all* of $\boldsymbol{\delta}^1_{2n+1}$, we also have

$$\begin{aligned}\varepsilon \in \mathcal{T} \Leftrightarrow \varepsilon \in \mathcal{WO}^* \wedge \forall m\, \forall s\, \forall \delta \{U^*(m, s, \varepsilon, \delta)\\ \Rightarrow [\forall \alpha_0 \cdots \forall \alpha_{m-1} [\alpha_0 \cdots \alpha_{m-1} \in \mathcal{P}^{2n+1}\\ \wedge \phi_i(\alpha_i) = |(R_\delta)_i|] \Rightarrow \exists \alpha\, \forall i < m((s)_i = \alpha(i) \wedge \phi_i(\alpha) = \phi_i(\alpha_i))]\},\end{aligned}$$

which demonstrates that $\mathcal{T} \in \Pi^1_{2n+2}$, assuming again the above estimate for V.

To verify that $V \in \Sigma^1_{2n+2}$ we use the following uniform version of Theorem 3.3.2 (which can be proved by the same argument): There is a recursive function f such that if $\varphi : W \twoheadrightarrow \kappa$, $\varphi' : W' \twoheadrightarrow \kappa'$ are $\boldsymbol{\Delta}^1_n$-norms on the $\boldsymbol{\Delta}^1_n$ sets W, W', with $\boldsymbol{\Delta}^1_n$-codes ε, ε' resp., then the relation $w \in W \wedge w' \in W' \wedge \varphi(w) \leq \varphi(w')$ is $\boldsymbol{\Delta}^1_{n+1}$ with $\boldsymbol{\Delta}^1_{n+1}$-code $f(\varepsilon, \varepsilon')$. Since for $\varepsilon \in \mathcal{WO}^*$, $|R_\varepsilon| = \operatorname{rank}(H_\varepsilon)$ and H_ε is a $\boldsymbol{\Delta}^1_{2n+1}$ prewellordering with $\boldsymbol{\Delta}^1_{2n+1}$-code $g(\varepsilon)$, where g is a recursive function, this version of Theorem 3.3.2 trivially implies that $V \in \Sigma^1_{2n+2}$.

The proof of (B) is now complete.

4.6. It remains so only to deal with (C) and this involves just a standard computation.

4.6.1. Lemma. *Let $A \subseteq \boldsymbol{\delta}^1_{2n+1}$ be such that $\tilde{A} \in \Delta$. Then*

$$\mathcal{L}(X) \Leftrightarrow X \subseteq \kappa \wedge X \in L[A]$$

is in Γ.

Proof. Since $\boldsymbol{\delta}^1_{2n+1}$ is a regular cardinal (see [13]) and $\kappa < \boldsymbol{\delta}^1_{2n-1}$, a standard collapsing argument shows that for an appropriate finite set of axioms ZF_N of ZF, including extensionality, we have for $X \subseteq \kappa$:

$$X \in L[A] \Leftrightarrow \exists \xi < \boldsymbol{\delta}^1_{2n+1}\, \exists \eta < \xi [X, \kappa \in L_\xi[A \cap \eta] \wedge L_\xi[A \cap \eta] \models \mathrm{ZF}_N].$$

Thus we have, since $L_\xi[A \cap \eta]$ as above has cardinality κ,

$$\begin{aligned}X \in L[A] \Leftrightarrow \exists M \subseteq \kappa\, \exists E \subseteq \kappa^2\, \exists x < \kappa\, \exists y < \kappa\, \exists z < \kappa\, \exists w < \kappa\\ \{E \subseteq M^2 \wedge x, y, z, w \in M\\ \wedge \langle M, E \rangle \models \text{`}\mathrm{ZF}_N \wedge V = L[z] \wedge y,\ w \in \mathrm{ORD} \wedge x \subseteq y \wedge z \subseteq w\text{'}\\ \wedge E \in \mathcal{WF} \wedge \forall t [tEw \Rightarrow (tEz \Leftrightarrow E(t) \in \tilde{A})]\\ \wedge |E(y)| = \kappa \wedge \forall u [uEy \Rightarrow (uEx \Rightarrow |E(u)| \in X)]\},\end{aligned}$$

where for $v \in M$, $E(v) = \{(p, q) \in M^2 : pEv \wedge qEv \wedge (pEq \vee p = q)\}$, so that for $\langle M, E \rangle$ as above, if $\langle M, E \rangle \models v \in \mathrm{ORD}$, then $E(v) \in \mathcal{WO}$. This clearly shows that $\mathcal{L}(X)$ is in Γ and we are done.

We have now completed the proof, as

$$\alpha \in L[T^{2n+1}] \Leftrightarrow \exists \varepsilon[\varepsilon \in CD^1 \wedge \mathcal{L}^*(\varepsilon) \wedge R_\varepsilon = \{\langle n, m\rangle : \alpha(n) = m\}],$$

so that $\omega^\omega \cap L[T^{2n+1}]$ is Σ^1_{2n+2}.

4.7. As a corollary of the theorem, we have that $\omega^\omega \cap L[T^{2n+1}]$ is independent of the choice of $\mathcal{P}^{2n+1}$, $\{\phi_m\}$ that are used to define T^{2n+1}. However, it is still open if the model $L[T^{2n+1}]$ itself is equally invariant (see [9, Appendix]).

PART II: Full ordinal games

5. The determinacy of ordinal games

5.1. Let κ be an ordinal. To each set $A \subseteq \kappa^\omega \times \kappa^\omega$, we associate the usual game $G(A; \kappa)$ played as follows:

I	II	
		Players I, II play alternatively
ξ_0		
	η_0	$\xi_0, \eta_0, \ldots$ from κ
ξ_1		and I wins iff
	η_1	
$\vdots$		$(\vec{\xi}, \vec{\eta}) \in A$.
$\vec{\xi}$	$\vec{\eta}$	

We shall study in this part of the paper questions related to the determinacy of such games.

It has been noted by Mycielski [14], in the early days of determinacy, that even *without using* AC, not all games $G(A; \omega_1)$ are determined. The proof is by contradiction and, as pointed out to one of the authors by H. Becker, it does not exhibit a particular example of a nondetermined such game: If all $G(A; \omega_1)$ are determined, in particular AD holds, so there are no uncountable wellorderings of sets of reals. Consider, however, the following game on ω_1:

I	II	
		II wins iff $\vec{\eta} = w \in \omega^\omega$ and
$\vec{\xi}$	$\vec{\eta}$	
		w codes a wellordering of
		ω of rank ξ_0.

Clearly I cannot have a winning strategy in this game. But if II has a winning strategy G and $F(\xi) = G(\xi, 0, 0, \ldots)$, then for each $\xi < \omega_1$, $F(\xi) = w$ is a code of ξ, so we have an uncountable wellordered set of reals, a contradiction.

When $\kappa < \Theta$ and $\phi : W \twoheadrightarrow \kappa$ is a norm on $W \subseteq \omega^\omega$, one can simulate the game

$G(A;\kappa)$ by a game $G^*_\phi(A, \omega^\omega)$ on ω^ω, its *coded version*. The game $G^*_\phi(A;\omega^\omega)$ is played as follows:

I	II	
		Players I, II alternatively
u^0_{I}		
	u^0_{II}	play $u^0_{\mathrm{I}}, u^0_{\mathrm{II}}, u^1_{\mathrm{I}}, u^1_{\mathrm{II}}, \ldots$ in ω^ω.
u^1_{I}		
	u^1_{II}	I wins iff
$\vdots$		
$\vec{u}_{\mathrm{I}}$	$\vec{u}_{\mathrm{II}}$	(i) For some i, u^i_{I} or u^i_{II} is not in W and for the least such i, say i_0, $u^{i_0}_{\mathrm{I}} \in W$ or (ii) For all i, u^i_{I} and u^i_{II} are in W and $(|\vec{u}_{\mathrm{I}}|, |\vec{u}_{\mathrm{II}}|) \in A$, where $|\vec{u}| = (|u^0|, |u^1|, \ldots)$ and $|u| = \phi(u)$.

An obvious but basic observation is that *using AC*, the two games $G(A;\kappa)$ and $G^*_\phi(A;\omega^\omega)$ are equivalent. This is because from AC, there is a function $F:\kappa \to \omega^\omega$, which picks for each $\xi < \kappa$ exactly one code $w = F(\xi) \in W$ for ξ. Thus, in the real world of ZFC, the question of the determinacy of $G(A;\kappa)$ is reduced to that of $G^*_\phi(A;\omega^\omega)$. This is of course a game on ω^ω, a more complicated space than κ. Our main result, however, will show that under cetain circumstances, depending just on the complexity of ϕ as in Part I, this game can be in turn simulated and reduced to a game $HG(A;\omega)$ on ω, so that various standard determinacy hypotheses yield the determinacy of $G^*(A;\omega^\omega)$ and thus $G(A;\kappa)$. This explains why one is apparently unable for instance to produce particular examples of undetermined games $G(A;\omega_1)$ on ω_1.

5.2. Before we embark on discussing these matters in detail, let us generalize a bit the basic context as in Part I. So let $\sim$ be an equivalence relation on a set $W \subseteq 2^\omega$. (Replacing ω^ω by 2^ω, is just for technical convenience in this section and is by no means necessary.) As usual, we denote by $|u|$ the equivalence class of $u \in W$ and by K the set of equivalence classes of $\sim$. If $A \subseteq K^\omega \times K^\omega$, the game $G(A;K)$ is defined exactly as before and so is also its coded version $G^*_\sim(A;2^\omega) \equiv G^*(A;2^\omega)$; thus $G^*_\phi \equiv G^*_{\sim_\phi}$.

The game HG is defined modulo a coding system $\mathscr{P}$ for perfect binary trees (recall Section 1.3.1 here). Let us introduce some notational conventions to facilitate its description:

Recall that $\langle a, c\rangle$ is good (relative to $\mathscr{P}$) if $a \in P \wedge c \in [\mathscr{P}_a]$. If $\langle a^0, c^0\rangle$ is good, define inductively $\langle u^i, a^{i+1}, c^{i+1}\rangle$ by

$$\langle u^i, a^{i+1}, c^{i+1}\rangle = (a^i)^*(c^i),$$

provided that $\langle a^i, c^i\rangle$ is good; otherwise $\langle u^i, a^{i+1}, c^{i+1}\rangle$ is not defined. Here $\langle a, b\rangle$,

$\langle a, b, c\rangle, \ldots$ refer to the standard recursive homeomorphisms of $2^\omega \times 2^\omega$, $2^\omega \times 2^\omega \times 2^\omega, \ldots$ with 2^ω.

The game $HG_{\sim,\mathcal{P}}(A; \omega) \equiv HG(A; \omega)$ is now played as follows:

I	II	
$\langle a_{\mathrm{I}}^0, c_{\mathrm{I}}^0\rangle$	$\langle a_{\mathrm{II}}^0, c_{\mathrm{II}}^0\rangle$	Players I, II play, bit by bit, respectively $\langle a_{\mathrm{I}}^0, c_{\mathrm{I}}^0\rangle$, $\langle a_{\mathrm{II}}^0, c_{\mathrm{II}}^0\rangle$ (so that this is actually a game on $\{0, 1\}$) and I wins iff (i) There is $i \geq 0$ such that either ($\langle a_{\mathrm{I}}^i, c_{\mathrm{I}}^i\rangle$ is not good $\vee\, u_{\mathrm{I}}^i \notin W$) or ($\langle a_{\mathrm{II}}^i, c_{\mathrm{II}}^i\rangle$ is not good $\vee\, u_{\mathrm{II}}^i \notin W$) and for the least such i, say i_0, $\langle a_{\mathrm{I}}^{i_0}, c_{\mathrm{I}}^{i_0}\rangle$ is good and $u_{\mathrm{I}}^{i_0} \in W$. or (ii) For all i, $\langle a_{\mathrm{I}}^i, c_{\mathrm{I}}^i\rangle$ and $\langle a_{\mathrm{II}}^i, c_{\mathrm{II}}^i\rangle$ are good and $u_{\mathrm{I}}^i, u_{\mathrm{II}}^i \in W$ and $(|\vec{u}_{\mathrm{I}}|, |\vec{u}_{\mathrm{II}}|) \in A$.

Our main result is now as follows (recall Lemma 1.4.2 here).

5.2.1. Theorem. *Let $\sim$ be a coarse equivalence relation on $W \subseteq 2^\omega$ and assume that $\sim \in \boldsymbol{\Gamma}$, where Γ is a category-adequate Spector pointclass. Then there is a coding system $\mathcal{P}$ for perfect binary trees such that for all $A \subseteq K^\omega \times K^\omega$ we have:*

$$\text{I(II) has a winning strategy in } HG_{\sim,\mathcal{P}}(A;\omega)$$
$$\Rightarrow \text{I(II) has a winning strategy in } G^*_{\sim}(A; 2^\omega).$$

The first result on the determinacy of ordinal games was proved by Harrington [2], who demonstrated the determinacy of all $G^*_\phi(A; 2^\omega)$ from AD for a particular norm $\phi : \omega^\omega \twoheadrightarrow \aleph_\omega$, by proving a result as above for this case. His basic technique, whose key elements were the use of perfect sets and the Recursion Theorem, then evolved into the form presented in this paper, with offspring the method used in Section 1.

5.3. Proof. From our hypotheses and Lemma 1.4.2, we know that there is a coding system $\mathcal{P}$ for perfect binary trees with $(\sim, \mathcal{P}, \Gamma)$ nice. Fix this $\mathcal{P}$ from now on.

Assume that player I has a winning strategy σ in $HG_{\sim,\mathcal{P}}(A; \omega)$. We shall prove then that I has also a winning strategy in $G^*_{\sim}(A: 2^\omega)$. (The case of player II is similar.)

Claim. *There is $f : (2^\omega)^{<\omega} \to (2^\omega \times 2^\omega)^{<\omega}$, such that*

(i) $t \subseteq t' \Rightarrow f(t) \subseteq f(t')$,

(ii) $\text{length}(f(t)) = \text{length}(t) + 1$,

(iii) if $u_0, \ldots, u_n \in W$ *and* $f(u_0, \ldots, u_n) = (a^0, v_0, a^1, v_1, \ldots, a^{n+1}, v_{n+1})$,

then we have

(a) $\forall i \leq n+1 (v_i \in W \wedge a^i \in P)$,

(b) *for any given* $c^{n+1} \in [\mathcal{P}_{a^{n+1}}]$ *let* $c^0, c^1, \ldots, c^n$ *be the uniquely determined members of* $[\mathcal{P}_{a^0}], \ldots, [\mathcal{P}_{a^n}]$ *resp. such that* $(a^i)^*(c^i) = \langle u_i, a^{i+1}, c^{i+1}\rangle$, *for* $i \leq n$. *Put*

$\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle = \langle a^0, c^0\rangle$ *and* $\langle a^0_{\mathrm{I}}, c^0_{\mathrm{I}}\rangle = \sigma(\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle)$; *then* $|u^i_{\mathrm{I}}| = |v_i|$ *for all* $i \leq n+1$. *(Notice here that since* $\langle a^{i+1}_{\mathrm{II}}, c^{i+1}_{\mathrm{II}}\rangle = \langle a^{i+1}, c^{i+1}\rangle$ *is good and* $u^i_{\mathrm{II}} = u_i \in W$ *for all* $i \leq n$, *we must have that* $\langle a^{i+1}_{\mathrm{I}}, c^{i+1}_{\mathrm{I}}\rangle$ *is good for all* $i \leq n$, *thus* $\langle u^{i+1}_{\mathrm{I}}, a^{i+2}_{\mathrm{I}}, c^{i+2}_{\mathrm{I}}\rangle$ *is defined for all* $i \leq n$ *and* $u^i_{\mathrm{I}} \in W$ *for all* $i \leq n+1$).

Granting this claim, we can obtain a strategy for I in $G^*(A; 2^\omega)$ as follows: When II plays $u_0, \ldots, u_n$, I answers by $v_0 \cdots v_{n+1}$, where $f(u_0 \cdots u_n) = (a^0, v_0, a^1, v_1, \ldots, a^{n+1}, v_{n+1})$.

We will show that this is indeed a winning strategy for player I. Assume II has played $u_0, u_1, \ldots$ in a run of the game and I produced $v_0, v_1, \ldots$ following this strategy. If for some i, $u_i \notin W$ and i_0 is the least such, then $u_0 \cdots u_{i_0-1} \in W$, so by (iii) above $v_0, \ldots, v_{i_0} \in W$, so already I won. So assume without loss of generality that $u_0, u_1, \ldots \in W$. Then also $v_0, v_1, \ldots \in W$ and $a^0, a^1, \ldots$ are produced as above with $a^0, a^1, \ldots \in P$. For each n now define perfect sets $C^n_0, C^n_1, \ldots, C^n_n \subseteq 2^\omega$ with $C^n_i \subseteq \mathscr{P}_{a^i}$ for all $i \leq n$, as follows:

$$C^n_n = \{c^n \in [\mathscr{P}_{a^n}]: (a^n)^*(c^n) = \langle u_n, a^{n+1}, x\rangle \text{ for some } x\},$$

$$C^n_{n-1} = \{c^{n-1} \in [\mathscr{P}_{a^{n-1}}]: (a^{n-1})^*(c^{n-1}) = \langle u_{n-1}, a^n, x\rangle \text{ for some } x \in C^n_n\},$$

$$\vdots$$

$$C^n_0 = \{c^0 \in [\mathscr{P}_{a^0}]: (a^0)^*(c^0) = \langle u_0, a^1, x\rangle \text{ for some } x \in C^n_1\}.$$

Note that

(a) $c^0 \in C^n_0 \Rightarrow \langle a^i, c^i\rangle$ are good for all $i \leq n$.

(b) $n' \geq n \Rightarrow C^{n'}_i \subseteq C^n_i$ for all $i \leq n$.

Thus $[\mathscr{P}_{a^0}] \supseteq C^0_0 \supseteq C^1_0 \supseteq C^2_0 \supseteq \cdots$, therefore $\bigcap_n C^n_0 \neq \emptyset$. Pick $c^0 \in \bigcap_n C^n_0$ and put $\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle = \langle a^0, c^0\rangle$. Then all $\langle a^i_{\mathrm{II}}, c^i_{\mathrm{II}}\rangle$ are good and $u^i_{\mathrm{II}} = u_i$. Moreover, if $\langle a^0_{\mathrm{I}}, c^0_{\mathrm{I}}\rangle = \sigma(\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle)$, then all $\langle a^i_{\mathrm{I}}, c^i_{\mathrm{I}}\rangle$ are good and all $u^i_{\mathrm{I}} \in W$. By (iii) of the claim, we have for all n (since $c^{n+1} \in [\mathscr{P}_{a^{n+1}}]$), that, $\forall i \leq n+1$, $|u^i_{\mathrm{I}}| = |v_i|$, therefore $|u^i_{\mathrm{I}}| = |v_i|$, for all i. But I won $HG(A; \omega)$ i.e. $(|\vec{u}_{\mathrm{I}}|, |\vec{u}_{\mathrm{II}}|) \in A$, so $(|\vec{v}|, |\vec{u}|) \in A$ and we are done.

Let us finally prove the claim. We shall construct $f(u_0 \cdots u_n)$ inductively on n. So we can assume that $f(u_0 \cdots u_{n-1}) = (a^0, v_0, \ldots, a^n, v_n)$ is known. We produce then the appropriate a^{n+1}, v_{n+1}. If one of the $u_0 \cdots u_n$ is not in W, let a^{n+1}, v_{n+1} be arbitrary. Otherwise define

$$H(a, c) = v,$$

where v is obtained as follows: Let $c^0, \ldots, c^n$ be the unique members of $[\mathscr{P}_{a^0}], \ldots, [\mathscr{P}_{a^n}]$ respectively such that $(a^i)^*(c^i) = \langle u_i, a^{i+1}, c^{i+1}\rangle$ for $i < n$ and $(a^n)^*(c^n) = \langle u_n, a, c\rangle$. Let $\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle = \langle a^0, c^0\rangle$ and $\langle a^0_{\mathrm{I}}, c^0_{\mathrm{I}}\rangle = \sigma(\langle a^0_{\mathrm{II}}, c^0_{\mathrm{II}}\rangle)$. Then note that $\{a^i_{\mathrm{I}}, c^i_{\mathrm{I}}\rangle$ is good and $u^i_{\mathrm{I}} \in W$ for $i \leq n+1$. Put $v = u^{n+1}_{\mathrm{I}}$.

Clearly $H: 2^\omega \times 2^\omega \to 2^\omega$ has range contained in W and is $\boldsymbol{\Gamma}$-measurable, by condition (**) of the niceness of $(\sim, \mathscr{P}, \Gamma)$ (see Definition 1.4.1). Thus by (*) of niceness, there is $a^{n+1} \in P$ such that for some fixed $\xi \in K$ and all $c^{n+1} \in [\mathscr{P}_{a^{n+1}}]$,

$|H(a^{n+1}, c^{n+1})| = \xi$. Let then $v_{n+1} = H(a^{n+1}, c_*^{n+1})$, where c_*^{n+1} is any member of $[\mathscr{P}_{a^{n+1}}]$, say the left most branch of $\mathscr{P}_{a^{n+1}}$.

The proof of Theorem 5.2.1 is now complete.

A careful inspection of the above argument gives also a key estimate for the complexity of winning strategies in the games $G^*(A; 2^\omega)$. A function $F: W^{<\omega} \to \omega^\omega$, where $W \subseteq \omega^\omega$, will be called $\boldsymbol{\Gamma}$-measurable, where Γ is a Spector pointclass, if $\{(\alpha_0 \cdots \alpha_n): \alpha_0 \cdots \alpha_n \in W \wedge F(\alpha_0 \cdots \alpha_n) \in N\}$ is in $\boldsymbol{\Gamma}$ for each open set N. This is as usual equivalent to saying that the relation

$$R(\alpha, t, n, m) \Leftrightarrow \forall i < t((\alpha)_i \in W) \wedge F((\alpha)_0 \cdots (\alpha)_{t-1})(n) = m,$$

is in $\boldsymbol{\Gamma}$.

5.3.1. Theorem. *Let $\sim$ be a coarse equivalence relation on $W \subseteq 2^\omega$ and assume that $\sim \in \boldsymbol{\Gamma}$, where Γ is a category-adequate Spector pointclass. Then there is a coding system $\mathscr{P}$ for perfect binary trees such that for all $A \subseteq K^\omega \times K^\omega$:*

I(II) *has a winning strategy in $HG_{\sim,\mathscr{P}}(A; \omega) \Rightarrow$* I(II) *has a winning strategy F in $G_\sim^*(A; 2^\omega)$ such that $F \restriction W^{<\omega}$ is $\boldsymbol{\Gamma}$-measurable.*

Note again that the complexity of F does not depend on A but only on $\sim$.

5.4. Let us put down now some obvious corollaries of Theorem 5.2.1. By $\kappa^{\mathbb{R}}$ we denote the supremum of the ranks of $\mathbf{HYP}(\mathbb{R})$ prewellorderings of ω^ω or equivalently the first admissible above $\mathbb{R}$ ordinal.

5.4.1. Corollary. *Let $\kappa \leq \kappa^{\mathbb{R}}$. Let $\phi: W \twoheadrightarrow \kappa$ be a* $\mathbf{IND}(\mathbb{R})$*-norm on the set $W \subseteq 2^\omega$, $W \in \mathbf{IND}(\mathbb{R})$. Then we have*

(i) AD$\Rightarrow$ *For all $A \subseteq \kappa^\omega \times \kappa^\omega$, $G_\phi^*(A; 2^\omega)$ is determined.*

(ii) *If all sets of reals which are definable from a countable sequence of ordinals are determined and AC holds, then for all $A \subseteq \kappa^\omega \times \kappa^\omega$, which are definable from a countable sequence of ordinals, $G(A; \kappa)$ is determined.*

Of course other, finer versions of this corollary, in which the exact level of the required determinacy hypotheses is computed, can be easily derived by tracing the steps of the proof of Theorem 5.2.1, but we will not make these explicit until needed in applications.

As $\Gamma = \mathrm{IND}(\mathbb{R})$ is the largest (except for its relativizations) known category-adequate Spector pointclass, $\kappa^{\mathbb{R}}$ is essentially the largest known at this time ordinal for which the conclusions of Corollary 5.4.1 can be proved (of course, e.g. (i) extends immediately to all ordinals of the same cardinality as $\kappa^{\mathbb{R}}$). We shall return to the open problem of extending this result to bigger κ's in Section 9.

6. Miscellaneous applications

We shall discuss here some applications of the ideas in Section 5, usually in the context of full AD.

6.1. First, let us consider a generalization of determinacy, considered by H. Friedman.

6.1.1. Definition. Let $\mathscr{F}$ be a collection of functions from ω^ω into ω^ω. By $\mathrm{AD}(\mathscr{F})$ we abbreviate the following statement: For $A \subseteq \omega^\omega \times \omega^\omega$, there is a function $f \in \mathscr{F}$ such that $\forall \beta\, A(f(\beta), \beta)$ or there is a function $g \in \mathscr{F}$ such that $\forall \alpha \neg A(\alpha, g(\alpha))$.

Clearly, if $\mathscr{F}_0$ is the class of all functions induced by strategies for either player I or II in the standard games on ω, then $\mathrm{AD} \Leftrightarrow \mathrm{AD}(\mathscr{F}_0)$. So $\mathrm{AD}(\mathscr{F})$ is an ostensibly weaker, at least when $\mathscr{F} \supseteq \mathscr{F}_0$, version of AD, asserting for each $A \subseteq \omega^\omega \times \omega^\omega$ the existence of a generalized (via $\mathscr{F}$) winning strategy for either I or II, in the game associated with A:

$$\begin{array}{ll} \mathrm{I} & \mathrm{II} \\ \alpha & \beta \end{array} \qquad \text{I wins iff } (\alpha, \beta) \in A.$$

H. Friedman asked if conversely AD(Continuous), AD(Borel), etc. imply AD. It was quickly shown that AD(Continuous) $\Leftrightarrow$ AD (Blass, Kunen, Mycielski). Then Kunen proved that

$(*)$ $\qquad \mathrm{AD(Borel)} \Rightarrow \mathrm{AD}$

and also

$(**)$ $\qquad \mathrm{AD}(\boldsymbol{\Delta}^1_2) \wedge \forall \alpha (\alpha^\# \text{ exists}) \Rightarrow \mathrm{AD}.$

As an immediate application of the method of proof of the main theorem in Section 5, we give below a general result, special cases of which include $(*)$, $(**)$. It is interesting to note here that Harrington [2] recognizes certain similarities between his arguments and those of Kunen in the proof $(*)$, which are earlier.

6.1.2. Theorem. *Let $\mathscr{F}$ be a class of functions from ω^ω into ω^ω and assume there is a category-adequate Spector pointclass Γ such that each function in $\mathscr{F}$ is $\boldsymbol{\Gamma}$-measurable. Then*

$$\mathrm{AD}(\mathscr{F}) \Rightarrow \mathrm{AD}.$$

Proof. Assume $\mathrm{AD}(\mathscr{F})$. Let $A \subseteq \omega^\omega \times \omega^\omega$. We want to show that the game $G(A; \omega)$ is determined, in the usual sense.

Consider the following norm $\phi : 2^\omega \twoheadrightarrow \omega$: $\phi(u) =$ least n such that $u(n) = 0$. It is of course enough to show that the game $G^*_\phi(A; 2^\omega)$ is determined. Let Γ be a category-adequate Spector pointclass such that every $f \in \mathscr{F}$ is $\boldsymbol{\Gamma}$-measurable. Clearly $\sim_\Phi \in \Gamma$, since $\sim_\Phi$ is arithmetical, so find $\mathscr{P}$, a coding system for perfect binary trees, with $(\sim_\Phi, \mathscr{P}, \Gamma)$ nice. Consider then the game $HG_{\sim_\Phi,\mathscr{P}}(A;\omega)$. By $\mathrm{AD}(\mathscr{F})$ it admits a strategy for I or II in $\mathscr{F}$ and therefore $\boldsymbol{\Gamma}$-measurable. But note that in the proof of Theorem 5.2.1 we never used the fact that the strategies in $HG_{\sim_\Phi,\mathscr{P}}(A;\omega)$ are of the standard type, but just that they are $\boldsymbol{\Gamma}$-measurable. Indeed, the only place in that argument where the strategy σ (for I say) comes in, is in the definition of H in the last two paragraphs of that proof and we only need there that H is $\boldsymbol{\Gamma}$-measurable. But for that the $\boldsymbol{\Gamma}$-measurability of σ is enough. So the proof of Theorem 5.2.1 goes through and establishes the determinacy of $G^*_\Phi(A;2^\omega)$.

For $\mathscr{F} = \text{Borel}$, we can clearly take $\Gamma = \Pi^1_1$ and so we have $(*)$. For $\mathscr{F} = \boldsymbol{\Delta}^1_2$, take $\Gamma = \Sigma^1_2$. By $\forall\alpha(\alpha^\#$ exists), Γ is category-adequate, so we have $(**)$. Actually $\forall\alpha(\alpha^\#$ exists) is not really needed because it is not hard to check that $\mathrm{AD}(\boldsymbol{\Delta}^1_2)$, instead of AD, is enough to carry out Kunen's argument that

$$\mathrm{AD} \Rightarrow \boldsymbol{\delta}^1_n \text{ is measurable},$$

when $n \geq 2$ (see [5, 5.1]). So $\mathrm{AD}(\boldsymbol{\Delta}^1_2) \Rightarrow \forall\alpha(\alpha^\#$ exists) and therefore

$$\mathrm{AD}(\boldsymbol{\Delta}^1_2) \Rightarrow \mathrm{AD}.$$

Further up, we have

$$\mathrm{AD}(\text{Projective}) + \mathrm{PD} \Rightarrow \mathrm{AD}$$

and also

$$\mathrm{AD}(\mathbf{HYP}(\mathbb{R})) + \mathrm{Det}(\mathbf{IND}(\mathbb{R})) \Rightarrow \mathrm{AD}$$

etc.

6.2. We turn now to another application of ordinal game determinacy. It is well-known that in the context of AD, ω_1 shares many large cardinal properties, for example it is measurable. Moreover from $\mathrm{AD}_{\mathbb{R}}$, Solovay [15] has shown that ω_1 is κ-supercompact for all $\kappa < \theta$. The question of whether this can be proved from AD only is still open, but our next result provides a partial answer. Although it is not clear yet what the supercompactness of ω_1 means in a choiceless universe, the particular way, given below, of producing fine, normal countably complete ultrafilters on $p_{\omega_1}(\kappa)$, can be useful in applications, as for example we shall see in the next section.

The proof of the result below is just Solovay's proof in the context of $\mathrm{AD}_{\mathbb{R}}$ mixed with Theorem 5.2.1.

6.2.1. Theorem. AD $\Rightarrow \omega_1$ *is* $\kappa^{\mathbb{R}}$*-supercompact.*

Proof. Put $\kappa = \kappa^{\mathbb{R}}$. We shall find a fine, normal, countably complete ultrafilter on $p_{\omega_1}(\kappa) = \{S \subseteq \kappa : \mathrm{card}(S) \leqslant \aleph_0\}$.

Fix $A \subseteq p_{\omega_1}(\kappa)$. Consider then the following game, G_A:

$$\begin{array}{lll} \mathrm{I} & \mathrm{II} & s_i \in p_\omega(\kappa) = \{S \subseteq \kappa : \mathrm{card}(S) < \aleph_0\}; \\ s_0 & & \text{II wins iff } \bigcup_i s_i \in A. \\ & s_1 & \\ s_2 & & \\ & s_3 & \\ \vdots & & \end{array}$$

We introduce first a coded version of this game. Fix an IND($\mathbb{R}$)-norm $\phi : W \twoheadrightarrow \kappa$ on $W \in \mathrm{IND}(\mathbb{R})$, $W \subseteq \omega^\omega$. Then define the following set of codes W^* for $p_\omega(\kappa)$:

$$\alpha \in W^* \Leftrightarrow \alpha(0) = 0 \vee [\alpha(0) > 0 \wedge \forall i < \alpha(0)((\alpha')_i \in W)],$$

where $\alpha' = \lambda t \cdot \alpha(t+1)$. For $\alpha \in W^*$, put

$$s_\alpha = \begin{cases} \emptyset & \text{if } \alpha(0) = 0, \\ \{|(\alpha')_i| : i < \alpha(0)\}, & \text{otherwise} \end{cases}$$

where $|\beta| = \phi(\beta)$. Thus $\alpha \mapsto s_\alpha$ maps W^* onto $p_\omega(\kappa)$. Put

$$\alpha \sim \beta \Leftrightarrow \alpha, \beta \in W^* \wedge s_\alpha = s_\beta.$$

Clearly $\sim$ is a coarse equivalence relation on W^* and $\sim \in \mathrm{IND}(\mathbb{R})$. Moreover $K_\sim$ can be identified with $p_\omega(\kappa)$ and s_α with the equivalence class of $\alpha \in W^*$, so that let G_A^* be the coded version of G_A defined as in Sections 5.1 and 5.2. By Section 5 G_A^* is determined.

Put now

$$A \in U \Leftrightarrow \text{II has a winning strategy in } G_A^*.$$

We shall prove that this U works.

Claim A. *U is a countably complete ultrafilter on $p_{\omega_1}(\kappa)$, which is fine i.e. for $\xi < \kappa$, $\hat{\xi} = \{S \in p_{\omega_1}(\kappa) : \xi \in S\} \in U$.*

Proof. If $A \in U$ and $B \supseteq A$, then clearly $B \in U$.

Suppose now that $A \notin U$. We shall then prove that $\sim A \in U$. Since $A \notin U$, I has a winning strategy $\mathcal{T}$ in G_A^*. The following is then a winning strategy for II in $G_{\sim A}^*$: I plays α_0, which we can assume is in W^*, otherwise he already lost. II(ignores this play and) plays $\alpha_1 = \mathcal{T}(\emptyset)$, i.e. the first move according to $\mathcal{T}$ in G_A^*. I plays now α_2, which we can again assume is in W^*. Let $\overset{*}{\cup}$ be a fixed total recursive function such that if $\alpha, \beta \in W^*$, then $\alpha \overset{*}{\cup} \beta \in W^*$ and $s_{\alpha \overset{*}{\cup} \beta} = s_\alpha \cup s_\beta$. II then plays

$\alpha_3 = \mathcal{T}(\alpha_0 \overset{*}{\cup} \alpha_2)$. I further plays $\alpha_4 \in W^*$. Then II plays $\alpha_5 = \mathcal{T}(\alpha_0 \overset{*}{\cup} \alpha_2, \alpha_4)$ etc., i.e. he pretends from now on that II plays successively $\alpha_0 \overset{*}{\cup} \alpha_2, \alpha_4, \alpha_6, \ldots$ and he follows $\mathcal{T}$.

We prove now countable completeness for U: Suppose $A_n \in U$, $n \in \omega$. Fix winning strategies $\mathcal{T}_n$ for II in $G^*_{A_n}$. Let $\langle\,\rangle$ be a 1–1 correspondence between $\omega \times \omega$ and ω such that $\langle n, m\rangle \geq \max\{n, m\} \wedge m < k \Rightarrow \langle n, m\rangle < \langle n, k\rangle$. The following is then a winning strategy for II *in* $G^*_{\bigcap_n A_n}$: In his $\langle n, m\rangle$th move he plays according to $\mathcal{T}_n$, treating all moves by either player since his $\langle n, m-1\rangle$th move as a single play of I, in the sense that if $\alpha_i, \alpha_{i+1}, \ldots, \alpha_r$ are these moves, then II pretends that I has played $F(\alpha_i, \ldots, \alpha_r)$, where $F: (\omega^\omega)^{<\omega} \to \omega^\omega$ is a recursive function such that if $\beta_1 \cdots \beta_l$ are in W^*, then $F(\beta_1 \cdots \beta_l) \in W^*$ and $s_{F(\beta_1, \ldots, \beta_l)} = \bigcup_{i=1}^l s_{\beta_i}$.

The proof that $\hat{\xi} \in U$ is obvious.

Claim B. *U is normal.*

Proof. Let $\{A_\xi\}_{\xi<\kappa}$ be a sequence of elements of U. We have to show that $\Delta_{\xi<\kappa} A_\xi = \{S \in p_{\omega_1}(\kappa) : \forall \xi \in S (S \in A_\xi)\} \in U$.

Consider first the game G below:

I	II	
ξ		$\xi, \eta < \kappa$; $s_i \in p_\omega(\kappa)$;
	η	II wins iff $\bigcup_i s_i \in A_\xi$,
s_0		
	s_1	
		(so η is irrelevant).
s_2		
	s_3	
$\vdots$		

Let G^* be the corresponding coded game. Clearly I does not have a winning strategy in it. So, since by Theorem 5.2.1 G^* is determined, II has a winning strategy in G^*. Thus there is a function $\mathcal{T}$ with domain W such that for each $w \in W$, $\mathcal{T}(w) \equiv \mathcal{T}_w$ is a winning strategy for II in $G^*_{A_{|w|}}$.

Let also H be a total recursive function such that if $\alpha \in W^*$, then $H(\alpha) = \langle w_0 \cdots w_{n-1}\rangle$ with $w_0 \cdots w_{n-1} \in W$ and $s_\alpha = \{|w_i| : i < n\}$.

We describe now a winning strategy for II in $G^*_{\Delta A_\xi}$: Assume that I has played $\alpha_0, \ldots, \alpha_{2i}$ and that II's moves $\alpha_1, \ldots, \alpha_{2i-1}$ have been inductively determined. We shall then define α_{2i+1}. We can assume also that all the reals $\alpha_0, \alpha_1, \ldots, \alpha_{2i-1}$, α_{2i} are in W^*. Let

$$H(\alpha_0 \overset{*}{\cup} \alpha_1 \overset{*}{\cup} \cdots \overset{*}{\cup} \alpha_{2i}) = \langle w_0 \cdots w_{n_i-1}\rangle,$$

where $\alpha \overset{*}{\cup} \beta \overset{*}{\cup} \gamma = (\alpha \overset{*}{\cup} \beta) \overset{*}{\cup} \gamma$ etc. Split $\{\langle i, m\rangle : m \in \omega\}$ into n_i infinite pieces $E_0^i, \ldots, E_{n_i-1}^i$ and let $E_j^i = \{k_0^{i,j}, k_1^{i,j}, \ldots\}$ in increasing order. II resolves from now on to play in his $k_m^{i,j}$th move according to $\mathcal{T}_{w_j}$ treating all the intermediate moves of both players, since his $k_{m-1}^{i,j}$th move as a single move of player I, as in the proof of Claim A. This (inductively) determines α_{2i+1}, since $i = \langle l, m\rangle$ for some $l \leq i$.

Let $\alpha_0, \alpha_1, \ldots$ be a run of the game, where II followed this strategy and assume that $\alpha_0, \alpha_1, \ldots \in W^*$. Let $S = \bigcup_m s_{\alpha_m}$. For each $\xi \in S$, we have to show that $S \in A_\xi$. Now $\xi \in s_{\alpha_m}$ for some m, thus for some large enough i, $G(\alpha_0 \overset{*}{\cup} \alpha_1 \overset{*}{\cup} \cdots \overset{*}{\cup} \alpha_{2i}) = \langle w_0 \cdots w_{n_i-1}\rangle$ and $\xi = |w_j|$, for some $j < n_i$. II has played according to the above instructions infinitely often according to $\mathcal{T}_{w_j}$, so that $s \in A_{|w_j|} = A_\xi$ and we are done.

6.3. Our final application deals with another problem related to the structure of cardinals under AD. What is at stake is the truth or falsity of the following conjecture: AD $\Rightarrow$ All regular cardinals $< \Theta$ are measurable. The result below, due to the authors and Moschovakis, provides some positive evidence.

6.3.1. Theorem. *Let M be an inner model of* ZF+DC+AD *with $\omega^\omega \subseteq M$. If* AC *holds and $\kappa \leq \kappa^{\mathbb{R}}$ is regular, then $M \vDash \kappa$ is measurable.*

So, for example, a possible scenario for an attempt to prove the above conjecture, for $\kappa \leq \kappa^{\mathbb{R}}$ at least, would be this: Extend generically the universe V of ZF+DC+AD to a model $V[G]$ of ZFC, without adding any new reals or destroying the regularity of κ. That such a possibility is not too far fetched is supported somewhat by recent developments in forcing techniques due to Steel [16].

Proof. Fix $\kappa \leq \kappa^{\mathbb{R}}$, κ regular. Let $\phi : W \twoheadrightarrow \kappa$ be an **IND**($\mathbb{R}$)-norm on the **IND**($\mathbb{R}$) set $W \subseteq \omega^\omega$. To each $A \subseteq \kappa$ associate now the following game, originally considered by Solovay, which we shall denote by J_A:

I	II	
		$\xi_i < \kappa$;
ξ_0		
	ξ_1	II wins iff $\sup_i \xi_i \in A$.
ξ_2		
	ξ_3	
$\vdots$		

Let $E_\omega = \{\xi < \kappa : \mathrm{cof}(\xi) = \omega\}$. Then the following is easy to prove by standard arguments using the regularity of κ: I(II) has a winning strategy in $J_A \Rightarrow A(\sim A)$ contains $C \cap E_\omega$, for some closed unbounded $C \subseteq \kappa$.

Let also J_A^* be the coded version of J_A, via ϕ. If $A \subseteq \kappa$ is in M, then since $M \vDash$AD, we have by Theorem 5.2.1 that $M \vDash J_A^*$ is determined. By the fact that

$\omega^\omega \subseteq M$, we also have that J_A^* is determined in the real world and so, by AC, J_A is determined, thus either A or $\sim A$ contains $C \cap E_\omega$ for some $C \subseteq \kappa$ closed unbounded in κ.

Working now *in M*, define for $A \subseteq \kappa$:

$$A \in U \Leftrightarrow \text{I has a winning strategy in } J_A^*.$$

Then by the above we have for each $A \in M$:

$$\begin{aligned} M \models A \in U \Leftrightarrow M \models \text{I} & \text{ has a winning strategy in } J_A^* \\ \Leftrightarrow & \text{ I has a winning strategy in } J_A^* \\ \Leftrightarrow & \text{ I has a winning strategy in } J_A \\ \Leftrightarrow & \exists C (C \subseteq \kappa \text{ is closed unbounded and } C \cap E_\omega \subseteq A). \end{aligned}$$

Thus $M \models$ 'U is a κ-complete nonprincipal ultrafilter on κ', and we are done.

7. Replacing AD by PD

7.1. The rather tightly controlled simulation of ordinal games and their coded versions by games on ω, that was developed in Section 5, can be used to generate techniques for proving *from PD only* results about projective sets, known to follow from much stronger determinacy hypotheses, like Det($P(\omega^\omega) \cap L(\omega^\omega]$) or Det($\mathbf{HYP}(\mathbb{R})$). We discuss in this section such a new proof for the result in Section 4 that $\omega^\omega \cap L[T^{2n+1}] = C_{2n+2}$. And we conclude in the next section by considering 'projective analogs' of certain standard set theoretical notions, which also provide an alternative method of 'translating' some proofs from AD to PD.

7.2. We state first the exact statement we want to prove.

7.2.1. Theorem. *Assume* Det($\boldsymbol{\Delta}^1_{2n+2}$), *when* $n \geq 1$. *Then* $\omega^\omega \cap L[T^{2n+1}] = C_{2n+2}$.

Proof. Let $\mathscr{P}$ be a complete Π^1_{2n+1} set and $\{\phi_m\}$ a Π^1_{2n+1}-scale on $\mathscr{P}$, $\phi_m : \mathscr{P} \twoheadrightarrow \delta^1_{2n+1} \equiv \kappa$. Recall that

$$T^{2n+1} = \{(\alpha(0), \phi_0(\alpha), \ldots, \alpha(m-1), \phi_{m-1}(\alpha)) : m \in \omega, \alpha \in \mathscr{P}\}.$$

Fix also a Π^1_{2n+1}-norm $\sigma : W \dashrightarrow \kappa$ on a complete Π^1_{2n+1} set W. The following lemma can be easily established and we omit the details.

Lemma A. *There is a tuple coding function* $\langle\,\rangle : (\omega \times \kappa)^{<\omega} \rightarrowtail\!\!\!\twoheadrightarrow \kappa$ *such that the following relation*

$$\begin{aligned} P(\alpha, t, \beta) \Leftrightarrow \alpha \in W \wedge \forall i < lh(t)[(\beta)_i \in W] \wedge \sigma(\alpha) \\ = \langle (t)_0, \sigma((\beta)_0), \ldots, (t)_{lh(t)-1}, \sigma((\beta)_{lh(t)-1}) \rangle \end{aligned}$$

is Δ^1_{2n+2}. *Moreover, for each* $A \subseteq \kappa^{<\omega}$, *if* $\langle A\rangle = \{\langle \xi_0 \cdots \xi_{m-1}\rangle : (\xi_0 \cdots \xi_{m-1}) \in A\} \subseteq \kappa$, *then* $L[A] = L[\langle A\rangle]$.

The next computation is similar to that of 4.5.2. Put $T = \langle T^{2n+1}\rangle$.

Lemma B. *If for each* $A \subseteq \kappa$, $A^* = \{x \in W : \sigma(x) \in A\}$, *then* $T^* \in \Delta^1_{2n+2}$.

Proof. Put

$$S(t, \beta) \Leftrightarrow \forall i < lh(t)((\beta)_i \in W) \wedge \exists \alpha \in \mathcal{P}[\forall i < lh(t)(\alpha(i) = (t)_i) \wedge \phi_i(\alpha) = \sigma((\beta)_i)].$$

Then

$$\alpha \in T^* \Leftrightarrow \exists t\, \exists \beta[P(\alpha, t, \beta) \wedge S(t, \beta)]$$
$$\Leftrightarrow \forall t\, \forall \beta[P(\alpha, t, \beta) \Rightarrow S(t, \beta)].$$

Thus it is enough to prove that $S \in \Delta^1_{2n+2}$. Note that since each ϕ_m maps $\mathcal{P}$ *onto* κ, we also have,

$$S(t, \beta) \Leftrightarrow \forall i < lh(t)((\beta)_i \in W)$$
$$\wedge \forall \alpha_0 \cdots \alpha_{lh(t)-1} \in \mathcal{P}[\phi_0(\alpha_0) = \sigma((\beta)_0)$$
$$\wedge \cdots \wedge \phi_{lh(t)-1}(\alpha_{lh(t)-1}) = \sigma(\beta_{lh(t)-1})$$
$$\Rightarrow \exists \alpha\, \forall i < lh(t)[\alpha(i) = (t)_i \wedge \phi_i(\alpha) = \phi_i(\alpha_i)]].$$

From these two expressions for S, it is clear that we only need to verify that the relation

$$\alpha \in \mathcal{P} \wedge \beta \in W \wedge \phi_i(\alpha) = \sigma(\beta)$$

is Δ^1_{2n+2}, which is clear by Theorem 3.3.2.

We will prove now that if $A \subseteq \kappa$ is such that $A^* \in \Delta^1_{2n+2}$, then $\omega^\omega \cap L[A]$ is contained in a countable Σ^1_{2n+2} set. (By a simple trick of Moschovakis [12], this can be extended also to $A \subseteq \kappa$, for which $A^* \in \Sigma^1_{2n+2}$: Indeed let $x \in A^* \Leftrightarrow \exists \alpha\, P(x, \alpha)$, where P is Π^1_{2n+1} and let f be recursive such that $P(x, \alpha) \Leftrightarrow f(x, \alpha) \in W$. Then put $B(\xi, \vartheta) \Leftrightarrow \xi, \vartheta < \boldsymbol{\delta}^1_{2n+1} \wedge \exists x\, \exists \alpha(\sigma(x) = \xi \wedge |f(x, \alpha)| < \vartheta)$. Then $B^*(x, y) \Leftrightarrow x, y \in W \wedge B(\sigma(x), \sigma(y))$ is Δ^1_{2n+2} and $\xi \in A \Leftrightarrow \exists \vartheta B(\xi, \vartheta)$. So $L[A] \subseteq L[B]$.)

Our method for this computation is motivated by (i) the idea of 'countable approximations' as developed (in model theory) in Kueker [10] and especially Barwise [1] and (ii) the proof of Theorem 6.2.1. To start with let us associate with each set S an operation

$$X \to X^S$$

(the *Mostowski collapse*, modulo S), defined by $\in$-induction as follows:

$$X^S = \{x^s : x \in X \cap S\}.$$

In particular, if $B \subseteq \mathrm{ORD}$ and $S \subseteq \mathrm{ORD}$, we have that $B^s = \{\xi^S : \xi \in B \cap S\}$, where $\xi^S =$ order type of $\xi \cap S$.

Given now λ, an infinite ordinal and $B \subseteq \lambda$, $\alpha \in \omega^\omega$ consider the following game, $\mathrm{I}(\beta, \alpha : \lambda)$:

I	II	$s_i \in p_\omega(\lambda)$; $S = \bigcup_i s_i \in p_{\omega_1}(\lambda)$;
s_0		
	s_1	II wins iff $\alpha \in L[B^s]$.
s_2		
	s_3	
$\vdots$		

Lemma C. $\alpha \in L[B] \Rightarrow$ II *has a winning strategy in* $\mathrm{I}(B, \alpha; \lambda)$.

Proof. Let $\mu > \lambda$ be such that $\{\beta, \alpha\} \subseteq L_\mu[B]$ and $L_\mu[B] \models \mathrm{ZF}_N \wedge V = L[B]$, for some appropriate large enough finite set ZF_N of the axioms of ZF. Let $f_0, f_1, \ldots$ be an enumeration of the Skolem functions of $\langle L_\mu[B], \varepsilon, \{\alpha\}, \{B\}\rangle$. Say f_i has n_i arguments. Split the odd natural numbers into infinitely many infinite pieces $E_0, E_1, \ldots$ and define the following strategy for II in $\mathrm{I}(B, \alpha; \lambda)$: Suppose I has played $s_0, \ldots, s_{2i}$. Then II plays $s_{2i+1} = \{f_j(\xi_1 \cdots \xi_{n_j}) : \xi_1, \ldots, \xi_{n_j} \in \bigcup_{k \leq i} s_{2_k}\} \cap \lambda$, where the j is such that $2i+1 \in E_j$. Let $S = \bigcup_n s_n$. Clearly $S = M \cap \lambda$, where M is the Skolem hull of $\bigcup_k s_{2k}$. Since $M \prec L_\mu[B]$ and $\{\alpha, B\} \subseteq M$, if $\pi : M \rightarrowtail\!\!\!\rightarrow N$ is the transitive collapse, then $N = L_{\mu'}[B']$, where $B' = \pi(B) = \{\pi(\xi) : \xi \in B \cap M\} = \{\pi(\xi) : \xi \in B \cap S\} = B^s$. So, since also $\pi(\alpha) = \alpha$, we have $\alpha \in L[B^s]$ i.e. II won.

Our plan from now on will be this:

(i) Let $S(\alpha, x) \Leftrightarrow x \in W \wedge$ II has a winning strategy in $\mathrm{I}(A \cap \sigma(x), \alpha; \sigma(x))$. If κ is regular (as it is with full AD), then we know that: $\alpha \in L[A] \Rightarrow \exists \xi < \kappa(\alpha \in L_\xi[A])$. Here we shall prove an approximation to this, i.e. $\alpha \in L[A] \Rightarrow \exists x \in W S(\alpha, x)$.

(ii) We shall compute that S is Δ^1_{2n+2} and for each $x \in W$, $S^x = \{\alpha : S(\alpha, x)\}$ is countable, from which it will follow easily that $\{\alpha : \exists x S(\alpha, x)\}$ is countable and of course Σ^1_{2n+2}, thereby completing the proof.

We start with (i).

Lemma D. $\alpha \in L[A] \Rightarrow \exists x \in W S(\alpha, x)$.

Proof. By Lemma C, with $B = A$, $\lambda = \kappa$, we know that II has a winning strategy in $\mathrm{I}(A, \kappa; \alpha)$. Suppose we can find a strategy f for II in this game with the following boundedness property: If for each $n < \omega$, $\theta < \kappa$ we put

$$g_n(\theta) = \sup\{\max(s_{2n+1}) + 1 : s_{2n+1} = f(s_0, s_2, \ldots, s_{2n}) \wedge \forall i \leq n(\max(s_{2i}) < \theta)\},$$

Then $g_n(\theta)<\kappa$. Then if we let $g(\theta)=\sup_n\{\theta+1, g_n(\theta)+1\}$, clearly $g(\theta)<\kappa$ as $\mathrm{cof}(\kappa)>\omega$. Put $\theta_0=\sup\{g(\omega), g^2(\omega), g^3(\omega), \ldots\}$. Then $\theta_0<\kappa$ and if I plays $s_0, s_2, \ldots$ with $\max(s_{2n})<\theta_0$ and II follows f, then for each n, $\max(s_{2n+1})<\theta_0$. But this clearly implies that II has a winning strategy in $\mathrm{I}(A\cap\theta_0, \alpha; \theta_0)$ i.e. if $x_0\in W$ is such that $\sigma(x_0)=\theta_0$, then $S(\alpha, x_0)$ and we are done.

In order to define such an f, we look at the coded game $\mathrm{I}^*(A, \alpha; \omega^\omega)$, associated with $\mathrm{I}(A, \alpha; \kappa)$ as in the proof of Theorem 6.2.1. Granting AC, II has a winning strategy in $\mathrm{I}^*(A, \alpha; \omega^\omega)$ — see the remarks in Section 7.3. concerning a method for eliminating AC from this proof. Then it will be enough to produce a winning strategy F for II in I^* such that for each $n<\omega$, $\theta<\kappa$ there is $g_n^*(\theta)<\kappa$ such that, in the notation of the proof of Theorem 6.2.1, if $\alpha_0, \alpha_2, \ldots, \alpha_{2n}\in W^*$ and $\max(s_{\alpha_{2i}})<\theta$ for all $i\leq n$, then $\max(s_{\alpha_{2n+1}})<g_n^*(\theta)$.

To find such an F, we go one further step back and we look at the game on ω simulating I^* as in Theorem 5.2.1. This game is Δ^1_{2n+2}, so determined and thus II has a winning strategy in it. Then by Theorem 5.3.1. II has a winning strategy F in I^* such that $F\restriction(W^*)^{<\omega}$ is $\boldsymbol{\Pi}^1_{2n+1}$-measurable. The existence of the bounds $g_n^*(\theta)$ for this F is now just a matter of a routine boundedness argument for the norm $\sigma: W \twoheadrightarrow \kappa$.

(Note here that the existence of an f as above would be trivial — any f would do — if κ was regular.)

We consider now part (ii).

Lemma E. *S is Δ^1_{2n+2}.*

Proof. For $\xi<\kappa$, let $\mathrm{I}^*(A\cap\xi, \alpha; \omega^\omega)$ be the coded version of $\mathrm{I}(A\cap\xi, \alpha; \xi)$ as in Theorem 6.2.1, where $W_\xi=\{x\in W: \sigma(x)<\xi\}$ with $\sigma\restriction W_\xi$ serves for coding ordinals $<\xi$ (and the corresponding W_ξ^* for coding members of $p_\omega(\xi)$). If II has a winning strategy in $\mathrm{I}(A\cap\xi, \alpha; \xi)$, then by AC, II has a winning strategy in $\mathrm{I}^*(A\cap\xi, \alpha; \omega^\omega)$. Then by an argument similar to that of the preceding lemma, we conclude that II has a winning strategy F in $\mathrm{I}^*(A\cap\xi, \alpha; \omega^\omega)$ which is $\boldsymbol{\Delta}^1_{2n+1}$. Moreover, trivially $\{x: \sigma(x)<\xi\wedge\sigma(x)\in A\}$ is $\boldsymbol{\Delta}^1_{2n+2}$ and so by Theorem 2.1.3 it is actually $\boldsymbol{\Delta}^1_{2n+1}$. Thus letting for any $\sim_\sigma$-invariant $H(x)$, $H_*(\xi)\Leftrightarrow\exists x\in W[x\in H\wedge\sigma(x)=\xi]$, we have:

$$\begin{aligned} S(\alpha, x)\Leftrightarrow x\in W\wedge\exists H\subseteq\omega^\omega\{&H \text{ is } \boldsymbol{\Delta}^1_{2n+1}\wedge H\subseteq W_{\sigma(x)} \text{ is } \sim_\sigma\text{-invariant}\\ &\wedge H_*=A\cap\sigma(x)\wedge\exists F: (\omega^\omega)^{<\omega}\to\omega^\omega[F \text{ is } \boldsymbol{\Delta}^1_{2n+1}\\ &\wedge F \text{ is a winning strategy for II in } \mathrm{I}^*(H_*, \alpha; \omega^\omega)]\}. \end{aligned}$$

By using a nice coding system for $\boldsymbol{\Delta}^1_{2n+1}$ pointsets by reals as in the proof of Theorem 3.1.1, we compute easily that $S\in\Sigma^1_{2n+2}$.

Since the game $\mathrm{I}(A\cap\xi, \alpha; \xi)$ is determined by Theorem 5.2.1, we also have that $S(\alpha, x)\Leftrightarrow x\in W\wedge\neg$(I has a winning strategy in $\mathrm{I}(A\cap\sigma(x), \alpha; \sigma(x))$), so S is also Π^1_{2n+2} and we are done.

We conclude now the proof by establishing

Lemma F. *For each $x \in W$, $S^x = \{\alpha: S(\alpha, x)\}$ is countable. Consequently, $\{\alpha: \exists x \in W\, S(\alpha, x)\}$ is countable.*

Proof. To see that the first assertion implies the second, let for $\xi < \kappa$, $A_\xi = \{\alpha: S(\alpha, x)\}$, where $\sigma(x) = \xi$, so that each A_ξ is countable and $\{\alpha: \exists x\, S(\alpha, x)\} = \bigcup_{\xi<\kappa} A_\xi$. By Theorem (1A-1) of [4], it is enough to show that the prewellordering $\alpha \leqslant \beta \Leftrightarrow \alpha, \beta \in \bigcup_{\xi<\kappa} A_\xi \wedge$ least $\xi(\alpha \in A_\xi) \leqslant$ least $\xi(\beta \in A_\xi)$ is Σ^1_{2n+2}. But

$$\alpha \leqslant \beta \Leftrightarrow \exists x \in W\{S(\alpha, x) \wedge \forall y[\sigma(y) < \sigma(x) \Leftrightarrow \neg S(\alpha, y)]\},$$

so, by Theorem 3.1.1, $\leqslant$ is Σ^1_{2n+2} and we are done.

To prove now the first assertion, fix $x \in W$. Let $\sigma(x) = \xi < \kappa$. Then $S^x = \{\alpha$: II has a winning strategy in $\mathrm{I}(A \cap \xi, \alpha; \xi)\}$. To show that S^x is countable it is sufficient to verify that it carries a $\boldsymbol{\Sigma}^1_{2n+2}$ wellordering.

Indeed, given $\alpha, \beta \in S^x$ let $\alpha \leqslant^* \beta \Leftrightarrow$ II has a winning strategy in the following game:

I	II	
		$s_i \in p_\omega(\xi)$; $S = \bigcup_i s_i$;
s_0		II wins iff for $B = A \cap \xi$
	s_1	
s_2		$\alpha \in L[B^S] \wedge \beta \in L[B^S] \wedge$
	s_3	$\alpha \leqslant_{L[B^S]} \beta$.
$\vdots$		

That $\leqslant^*$ is $\boldsymbol{\Sigma}^1_{2n+2}$ follows as in the proof of Lemma E. That $\leqslant^*$ is reflexive is obvious. The verification of the other properties of a wellordering is easy, using the fact that the game defining $\leqslant^*$ is determined and arguments exactly like those in the proof of Theorem 6.2.1, so we omit the details.

7.3. Strictly speaking, we used above AC in deducing the determinacy of certain ordinal games from the determinancy of their coded versions. The following standard device allows us to eliminate these uses of AC: Assume ZF+DC+Det($\boldsymbol{\Delta}^1_{2n+2}$). Let $\mathbb{C}$ be the following notion of forcing

$$\mathbb{C} = \{f: \xi \to \omega^\omega \mid \xi < \omega_1\},$$

and let G be $\mathbb{C}$-generic over the universe V. Since $V \models \mathrm{DC}$, V and $V[G]$ have the same reals, so $V[G] \models \mathrm{ZF}+$'$\omega^\omega$ is wellorderable'$+\mathrm{Det}(\boldsymbol{\Delta}^1_{2n+2})$. Thus $V[G] \models C_{2n+2} = \omega^\omega \cap L[T^{2n+1}]$, so by absoluteness, $C_{2n+2} = \omega^\omega \cap L[T^{2n+1}]$.

7.4. Let us conclude by pointing out a further corollary of Theorem 7.2.1. Some time ago Solovay has shown that the assumption that $\omega^\omega \cap L[T^{2n+1}, \alpha]$ is countable, for all $\alpha \in \omega^\omega$, implies that every $\boldsymbol{\Sigma}^1_{2n+2}$ set is Ramsey. The proof uses Mathias forcing over these models, exactly as the analogous proof for $\mathbf{\Sigma}^1_2$ sets does relative to the models $L[\alpha]$. Since this assumption follows from strong forms of determinacy, like $\mathrm{Det}(P(\omega^\omega)\cap L[\omega^\omega])$, one has the fact that all projective sets are Ramsey. It was not known, however, if this followed from just PD. Clearly Theorem 7.2.1 fills this gap.

7.4.1 Corollary. *Assume* $\mathrm{Det}(\mathbf{\Delta}^1_{2n+2})$. *Then all* $\mathbf{\Sigma}^1_{2n}$ *sets are Ramsey.*

8. Projective set theory

8.1. The results in this paper, especially the ones in Sections 2, 3, can be also used to develop, working in PD only, 'projective analogs' of certain standard aspects of the theory of ordinals and cardinals. One first defines, in a more or less straightforward fashion, 'projective analogs' of the standard set theoretic notions of cardinality, regularity, measurability etc., and then proves, using only PD, 'projective analogs' of standard set theoretical facts concerning these notions. Finally, and most importantly for our purposes, several of the consequences of AD for the structure of cardinals are transferred to this 'projective context', using again only PD. Since many times when AD is used to prove a result about projective sets, it is only through these consequences, we have as a byproduct a method for replacing AD by PD in these proofs, which consists in verifying that the 'projective analogs' of these consequences suffice for the arguments. We give a couple of examples below, one of which is another computation of $\omega^\omega\cap L[T^{2n+1}]$ in PD (however, it is not clear that it can be carried out in $\mathrm{Det}(\mathbf{\Delta}^1_{2n+2})$ only, as in Section 7).

8.2. We start now the very brief and far from complete sketch of a development of 'projective set theory', especially concentrating on the 'projective analogs' of the results in the theory of projective ordinals from AD. For comparison, our presentation in the beginning, parallels roughly that of [5]. From now on and for the rest of this section, we assume without further explicit mentioning ZF+DC+PD *only*.

8.2.1. Definition. An ordinal λ is called *projective* or a *p-ordinal* if it is the rank of a projective prewellordering of ω^ω, i.e. $\lambda<\mathbf{\delta}^1_n$ for some n. If $A\subseteq\lambda^m$, then A is projective or a *p-set* if for some projective norm $\phi:\omega^\omega\twoheadrightarrow\lambda$, $A^*_\phi=\{(w_1\cdots w_m):(\phi(w_1)\cdots\phi(w_m))\in A\}$ is projective. By Theorem 3.3.2 this definition is intrinsic, i.e. independent of the choice of ϕ. A function $f:\lambda\to\mu$ where λ, μ are p-ordinals is a *p-function* iff its graph is a p-set. A sequence $\{A_\xi\}_{\xi<\lambda}$ of subsets of some product space $\mathscr{X}$, where λ is a p-ordinal, is a *p-sequence* if for some projective norm $\phi:\omega^\omega\twoheadrightarrow\lambda$, we have that

$$A(w,x)\Leftrightarrow x\in A_{\phi(w)}$$

is projective. And a sequence $\{A_\xi\}_{\xi<\lambda}$ of p-sets $A_\xi \subseteq \mu$, where λ, μ are p-ordinals is a *p-sequence* if for some projective norm $\sigma : \omega^\omega \twoheadrightarrow \mu$, the sequence $\{(A_\xi)^*_\sigma\}_{\xi<\lambda}$ is a p-sequence. Again these notions are intrinsic.

A p-ordinal κ is a *p-cardinal* if there is no $\kappa' < \kappa$ and p-function $f : \kappa' \twoheadrightarrow \kappa$. The *p-cofinality* of a p-ordinal κ, in symbols *p-cof*(κ) is the least $\kappa' \leqslant \kappa$ for which there is a cofinal p-function $g : \kappa' \to \kappa$; κ is *p-regular* iff p-cof$(\kappa) = \kappa$.

The proof (of Moschovakis) in [5, 2.2] that

$$\mathrm{AD} \Rightarrow \forall n \geqslant 1,\ \boldsymbol{\delta}^1_n \text{ is a cardinal},$$

is readily adopted to show that each $\boldsymbol{\delta}^1_n$ is a p-cardinal. Moreover, with a little more work the proofs (of Kechris, Kunen–Martin) in [5, 3.10, 3.12] that

$$\mathrm{AD} \Rightarrow \forall n \geqslant 1,\quad \boldsymbol{\delta}^1_{2n} = (\boldsymbol{\delta}^1_{2n-1})^+ \wedge \boldsymbol{\delta}^1_{2n-1} \text{ is the successor of a cardinal of cofinality } \omega,$$

can be transcribed to show that for each $n \geqslant 1$, $\boldsymbol{\delta}^1_{2n}$ is the least p-cardinal bigger than $\boldsymbol{\delta}^1_{2n-1}$ and $\boldsymbol{\delta}^1_{2n-1}$ is the least p-cardinal bigger than a p-cardinal of cofinality ω (note that p-cof$(\kappa) = \omega \Leftrightarrow \mathrm{cof}(\kappa) = \omega$). Finally, it is easy again to transfer from [5, 4.1] the proof (of Kunen) that

$$\mathrm{AD} \Rightarrow \forall n \geqslant 1,\quad \boldsymbol{\delta}^1_n \text{ is regular},$$

to show that each $\boldsymbol{\delta}^1_n$ is p-regular.

Let now $\pi_0, \pi_1, \pi_2, \ldots, \pi_\xi, \ldots (\xi < \rho)$ be the increasing enumeration of the p-cardinals. Thus $\pi_0 = \omega$, $\pi_1 = \omega_1$ and the arguments (of Kunen, Martin, Solovay) in [5, 8.4] that

$$\mathrm{AD} \Rightarrow \forall n \leqslant \omega,\quad u_n = \omega_n,$$

where $u_\xi = \xi$th uniform indiscernible, translate without difficulty to the fact that $\forall n \leqslant \omega$, $\pi_n = u_n$. Then Martin's proof [5, 6.4] that

$$\mathrm{AD} \Rightarrow \boldsymbol{\delta}^1_3 = \omega_{\omega+1},$$

gives that $\boldsymbol{\delta}^1_3 = \pi_{\omega+1}$ and so $\boldsymbol{\delta}^1_4 = \pi_{\omega+2}$. In general, $\boldsymbol{\delta}^1_{2n+1} = \pi_{\rho_n+1}$, $\boldsymbol{\delta}^1_{2n+2} = \pi_{\rho_n+2}$, where π_{ρ_n} is a p-cardinal of cofinality ω, so ρ_n is an ordinal of cofinality ω. The actual value of ρ_n is not known for $n \geqslant 2$, but work of Kunen suggests that $\rho_2 = \omega^3$ (here ω^3 denotes ordinal exponentiation). (Of course Kunen works in the context of AD and the above is just the natural translation to the PD context.)

8.3. Before we proceed, let us sketch an application which illustrates the ideas described until now in this section. We shall replace AD by PD in the proof of Theorem 3.2 in [6], which states that $\boldsymbol{\Sigma}^1_2$ equivalence relation on ω^ω has either $\leqslant\omega_1$ or $2^{\aleph_0}$ equivalence classes. (R. Sami has also found such a proof in PD by a different method earlier.)

For that, it is enough to prove the projective analog of the theorem in [6, Section 1] i.e. that if $\{A_\xi\}_{\xi<\lambda}$ is a p-sequence of $\boldsymbol{\Sigma}^1_{2n}$ sets, where λ is a p-ordinal, then $\bigcup_{\xi<\lambda} A_\xi$ is $\boldsymbol{\Sigma}^1_{2n}$. Tracing the steps of the argument in [6, Section 1] and taking $n=1$ for notational simplicity, let λ be the least p-ordinal for which there is a p-sequence $\{A_\xi\}_{\xi<\lambda}$ of $\boldsymbol{\Sigma}^1_2$ sets with $\bigcup_{\xi<\lambda} A_\xi \notin \boldsymbol{\Sigma}^1_2$, towards a contradiction. Then λ is a p-regular cardinal. (*Proof.* If $f:\lambda'\to\lambda$ is a cofinal p-function, with $\lambda'<\lambda$, put $B_{\xi'}=\bigcup_{\xi<f(\xi')}A_\xi$ for $\xi'<\lambda'$. Clearly, $\bigcup_{\xi<\lambda'}B_{\xi'}=\bigcup_{\xi<\lambda}A_\xi$, so it is enough to check that $\{B_{\xi'}\}_{\xi'<\lambda'}$ is a p-sequence. But if $\phi':\omega^\omega \twoheadrightarrow \lambda'$ is a projective-norm,

$$\begin{aligned}B(w',\alpha)&\Leftrightarrow \alpha\in B_{\phi'(w)}\Leftrightarrow \exists\xi<f(\phi'(w))(\alpha\in A_\xi)\\&\Leftrightarrow \exists w[\phi(w)<f(\phi'(w))\wedge\alpha\in A_{\phi(w)}],\end{aligned}$$

where $\phi:\omega^\omega\twoheadrightarrow\lambda$ is a projective norm and we are done.) Next we want to show that $\lambda\geqslant\boldsymbol{\delta}^1_3$. For that let

$$\Gamma=\left\{\bigcup_{\xi<\lambda}A_\xi\colon \{A_\xi\}_{\xi<\lambda} \text{ is a } p\text{-sequence and } \forall\xi<\lambda(A_\xi\in\boldsymbol{\Sigma}^1_2)\right\}.$$

Then $\boldsymbol{\Sigma}^1_2\subsetneq\Gamma$ and Γ is closed under continuous images and preimages. As every $A\in\Gamma$ is projective, Wadge's lemma implies that $\boldsymbol{\Pi}^1_2\subseteq\Gamma$ and so $\boldsymbol{\Sigma}^1_3\subseteq\Gamma$. But by a straightforward projective analog of Martin's result that 'AD$\Rightarrow\boldsymbol{\Delta}^1_3$ is closed under unions of sequences of $<\boldsymbol{\delta}^1_3$ sets', we have that if $\lambda<\boldsymbol{\delta}^1_3$, $\Gamma\subseteq\boldsymbol{\Delta}^1_3$, a contradiction.

Now since $\Gamma\supseteq\boldsymbol{\Sigma}^1_3$, let $\prec$ be an arbitrary $\boldsymbol{\Sigma}^1_3$ wellfounded relation and write $\prec=\bigcup_{\xi<\lambda}(\prec_\xi)$, where $\prec_\xi$ is a $\boldsymbol{\Sigma}^1_2$ wellfounded relation and by the minimality of λ we can assume that $\xi\leqslant\eta<\lambda\Rightarrow\prec_\xi\subseteq\prec_\eta$ (replace if necessary $\prec_\xi$ by $\bigcup_{\xi'\leqslant\xi}(\prec_{\xi'})$). For each $\alpha\in\mathrm{Field}(\prec)$, find $\xi<\lambda$ large enough so that for all $\eta\geqslant\xi$, $\alpha\in\mathrm{Field}(\prec_\eta)$. Then for $\eta\geqslant\xi$ let $|\alpha|_\eta=\mathrm{rank}_{\prec_\eta}(\alpha)$. By Kunen–Martin $|\alpha|_\eta<\boldsymbol{\delta}^1_2$, $\forall\eta\geqslant\xi$. Also $\xi\leqslant\eta\leqslant\eta'<\lambda\Rightarrow|\alpha|_\eta\leqslant|\alpha|_{\eta'}$. Since moreover by Theorem 3.3.2 $\eta\mapsto|\alpha|_\eta$ is a p-function and p-cof$(\lambda)=\lambda\geqslant\boldsymbol{\delta}^1_3>\boldsymbol{\delta}^1_2$ we have that $|\alpha|_\eta=\text{constant}=f(\alpha)<\boldsymbol{\delta}^1_2$ for all sufficiently large η. Again $\alpha\prec\beta\Rightarrow f(\alpha)<f(\beta)$, so $\mathrm{rank}(\prec)\leqslant\boldsymbol{\delta}^1_2$. As $\prec$ was arbitrary $\boldsymbol{\Sigma}^1_3$, $\boldsymbol{\delta}^1_3\leqslant\boldsymbol{\delta}^1_2$, a contradiction.

8.4. Let us consider now some large cardinal properties of projective cardinals, particularly measurability. Some terminology first.

8.4.1. Definition. Let λ be a p-ordinal, $\mathcal{U}\subseteq p-P(\lambda)=$ *projective power of* $\lambda=\{X\subseteq\lambda\colon X \text{ is a p-set}\}$. We want to define what it means for $\mathcal{U}$ to be a p-set. Fix $\phi:\omega^\omega\twoheadrightarrow\lambda$, a projective-norm. Say ϕ is a $\boldsymbol{\Delta}^1_n$-norm. Then by Theorem 2.1.3, for each p-set $A\subseteq\lambda$, A^* is $\boldsymbol{\Delta}^1_n$. Let $U(\varepsilon,\alpha)$ be Σ^1_n and universal for the $\boldsymbol{\Sigma}^1_n$ subsets of ω^ω and say that 'ε is a $\boldsymbol{\Delta}^1_n$-code' if $U_{(\varepsilon)_0}=\neg U_{(\varepsilon)_1}$. Put then $H_\varepsilon=U_{(\varepsilon)_0}\in\boldsymbol{\Delta}^1_n$. Finally let,

$$I=\{\varepsilon\colon \varepsilon \text{ is a } \boldsymbol{\Delta}^1_n\text{-code}\wedge H_\varepsilon \text{ is } \sim_\phi\text{-invariant}\},$$

and for $\varepsilon\in I$, put

$$A_\varepsilon=(H_\varepsilon)_*\subseteq\lambda.$$

Call now $\mathcal{U}$ a p-set iff

$$\mathcal{U}^* = \{\varepsilon \in I: A_\varepsilon \in \mathcal{U}\}$$

is projective. Again it is not hard to verify that this is an intrinsic notion, independent of ϕ, U, etc.

A p-cardinal κ is *p-measurable* if there is a p-set $\mathcal{U} \subseteq p - P(\kappa)$ with the following properties

(i) $\mathcal{U}$ is an ultrafilter on $p - P(\kappa)$,

(ii) $\{\xi\} \notin \mathcal{U}$, $\forall \xi < \kappa$.

(iii) If $\{A_\xi\}_{\xi<\lambda}$ is a p-sequence of subsets of κ with $\lambda < \kappa$, then we have:

$$\forall \xi < \lambda, \quad A_\xi \in \mathcal{U} \Rightarrow \bigcap_{\xi<\lambda} \in \mathcal{U}.$$

We shall call such a $\mathcal{U}$ a *p-measure* on κ. $\mathcal{U}$ is *p-normal* if for each p-function $f: \kappa \to \kappa$ with $\{\xi: f(\xi) < \xi\} \in \mathcal{U}$, there is $\xi_0 < \kappa$ such that $\{\xi: f(\xi) = \xi_0\} \in \mathcal{U}$. Equivalently, this means that for every p-sequence of elements of $\mathcal{U}$, $\{A_\xi\}_{\xi<\kappa}$, their diagonal intersection $\Delta_{\xi<\kappa} A_\xi \in \mathcal{U}$.

Kunen's proof in [5, 5.1] that AD$\Leftrightarrow \forall n \leqslant 1$, $\boldsymbol{\delta}^1_n$ is measurable, can be used to show that each $\boldsymbol{\delta}^1_n$ is p-measurable for each $n \geqslant 1$.

Proceeding further, we note that the usual argument shows that every p-measurable cardinal carries a p-normal measure. (Note here that every p-measure is closed under *arbitrary countable* intersections.) From that we can derive the projective analog of Rowbottom's proof that 'measurable $\Rightarrow$ Ramsey'. Let us give a sketch of this argument as a further example of the use of some ideas involved in recasting proofs in the projective context.

8.4.2. Theorem (PD). *Let κ be a p-measurable cardinal. Let $\mathcal{U}$ be a p-normal measure on κ. Then for each p-function $f: [\kappa]^n \to \mu$, where $n < \omega$, $\mu < \kappa$, there is $X \in \mathcal{U}$, X homogeneous for f.*

Proof. Fix $\phi: \omega^\omega \twoheadrightarrow \kappa$ a $\boldsymbol{\Delta}^1_m$-norm and define I, H_ε, $\mathcal{U}^*$ as in Definition 8.4.1. Let N be large enough so that $N > m$ and $\mathcal{U}^* \in \boldsymbol{\Delta}^1_N$. As usual we shall prove the theorem by induction on n. To carry out the induction step, we actually prove a stronger effective version of the result, namely the following:

Claim. *For each $n \geqslant 1$, there is a continuous function $h_n: \omega^\omega \to \omega^\omega$ such that, for every $f: [\kappa]^n \to \mu$, with $\mu < \kappa$, there is $X \subseteq \omega$, $X \in \mathcal{U}$, X homogeneous for f and such that for every δ which is a $\boldsymbol{\Delta}^1_N$-code of f, $h_n(\delta)$ is a $\boldsymbol{\Delta}^1_N$-code of X.*

(By a $\boldsymbol{\Delta}^1_N$-code of a p-set $R \subseteq \kappa^l$, we mean a $\boldsymbol{\Delta}^1_N$-code of R^*. Note that since $N > m$ each such R^* is $\boldsymbol{\Delta}^1_N$.)

The proof of the claim is by induction on n:

Case 1: $n=1$. Then $f:\kappa\to\mu$, $\mu<\kappa$. So there is unique $\xi_0<\mu$, with $X=\{\xi: f(\xi)=\xi_0\}\in\mathcal{U}$. Note now the following explicit definition of X^*, in terms of f:

$$\begin{aligned} w\in X^* &\Leftrightarrow \exists\varepsilon[\varepsilon\in\mathcal{U}^*\wedge f\upharpoonright A_\varepsilon \text{ is constant } \wedge w\in H_\varepsilon] \\ &\Leftrightarrow \forall\varepsilon[\varepsilon\in\mathcal{U}^*\wedge f\upharpoonright A_\varepsilon \text{ is constant} \\ &\Rightarrow \exists v\in H_\varepsilon(f(\sigma(w))=f(\sigma(v)))]. \end{aligned}$$

So a $\boldsymbol{\Delta}^1_N$-code for X can be effectively computed (i.e. via a continuous function) from a $\boldsymbol{\Delta}^1_N$-code of f.

Case 2: Assume $n>1$ and we have already proved the result for $1,\dots,n-1$. Consider now $f:[\kappa]^n\to\mu$, $\kappa<\mu$. Define then as usual for $\xi<\kappa$:

$$f_\xi(\xi_1\cdots\xi_{n-1})=\begin{cases} f(\xi,\xi,\dots,\xi_{n-1}) & \text{if } \xi<\xi_1<\cdot\cdot<\xi_{n-1},\\ 0 & \text{otherwise.}\end{cases}$$

Then $f_\xi:[\kappa]^{n-1}\to\mu$, so by induction hypothesis there is $X_\xi\in\mathcal{U}$, homogeneous for f_ξ and for every $\boldsymbol{\Delta}^1_N$-code δ of f and every w with $\phi(w)=\xi$, $h_{n-1}(g(\delta,w))$, is a $\boldsymbol{\Delta}^1_N$-code of X_ξ, where g is continuous, $g:(\omega^\omega)^2\to\omega^\omega$ and for each δ, w as above, $g(\delta,w)$ is a $\boldsymbol{\Delta}^1_N$-code of f_ξ. Let λ_ξ be the unique ordinal in $f''_\xi[X_\xi]^{n-1}$. Then if $\bar f(\xi)=\lambda_\xi$, we have also $\bar f:\kappa\to\mu$ and there is a continuous $\bar h:\omega^\omega\to\omega^\omega$, with $\bar h(\delta)$ a $\boldsymbol{\Delta}^1_N$-code of $\bar f$. This is because

$$\begin{aligned} \bar f(\xi)=\lambda &\Leftrightarrow \exists X\in\mathcal{U}[f_\xi\upharpoonright[X]^{n-1} \text{ is constant} \\ &\qquad \wedge\exists\bar x\in[X]^{n-1}(f_\xi(\bar x)=\lambda)] \\ &\Leftrightarrow \forall X\in\mathcal{U}[f_\xi\upharpoonright[X]^{n-1} \text{ is constant} \\ &\Rightarrow \exists\bar x\in[X]^{n-1}(f_\xi(\bar x)=\lambda)]. \end{aligned}$$

So by Case 1, let $\bar X$ be homogeneous for $\bar f$ with $\boldsymbol{\Delta}^1_N$-code $h_1(\bar h(\delta))$. Finally, put

$$\bar Y=\Delta_{\xi<\kappa}X_\xi\cap\bar X.$$

As usual $\bar Y$ is homogeneous for f and $\bar Y\in\mathcal{U}$, since the sequence $\{X_\xi\}_{\xi<\kappa}$ is a p-sequence (*Proof*: $v\in(X_{\phi(w)})^*\Leftrightarrow v$ belongs to the $\boldsymbol{\Delta}^1_N$-set coded by $h_{n-1}(g(\delta,w))$.) Moreover,

$$v\in\bar Y^*\Leftrightarrow\forall w[\phi(w)<\phi(v)\Rightarrow v\in(X_{\phi(w)})^*]\wedge v\in(\bar X)^*,$$

so we can effectively compute a $\boldsymbol{\Delta}^1_N$-code of Y from δ and we are done. Of course we are using heavily in these arguments the ordinal quantification results of Section 3.

8.5. Using now Theorem 8.4.2 it is not hard to prove that for every p-set $A\subseteq\lambda$, where λ is a p-ordinal, $A^\#$ exists and that all sufficiently large $\boldsymbol{\delta}^1_n$'s are Silver indiscernibles for $L[A]$, so regular cardinals in $L[A]$. (Note here that for any p-ordinal μ, every first-order definable relation on $\langle L_\mu[A], \varepsilon, A\rangle$ is a p-set.) As a corollary we have

8.5.1. Lemma. *Let $A \subseteq \boldsymbol{\delta}^1_n$ be a p-set. Then $\boldsymbol{\delta}^1_n$ is regular in $L[A]$ and so if $\alpha \in L[A]$, then $\alpha \in L_\xi[A \cap \eta]$, where $\eta < \xi < \delta^1_n$.*

Proof. If $L[A] \models$ '$\boldsymbol{\delta}^1_n$ is singular', there is $f : \lambda \to \delta^1_n$, cofinal, where $f \in L[A]$ and $\lambda < \boldsymbol{\delta}^1_n$. By the preceding remarks $f \in L_{\boldsymbol{\delta}^1_N}(A)$ for all sufficiently large N, so f is first-order definable with ordinal parameters in $\langle L_{\boldsymbol{\delta}^1_N}[A], \varepsilon, A\rangle$, so it is a p-function, contradicting the p-regularity of $\boldsymbol{\delta}^1_N$.

We finish now this section noting that it is at this stage routine, using Lemma 8.5.1, to translate the argument in Section 4 in the projective context, thereby providing another proof that $\omega^\omega \cap L[T^{2n+1}] = C_{n+2}$ in PD.

9. Some open problems

We collect here some questions related to the results presented in this paper.

9.1. First, it is clear that we would like to know if there are category-adequate Spector pointclasses Γ strictly containing IND($\mathbb{R}$) (i.e. IND($\mathbb{R}$) $\subsetneq \Delta$). What is difficult is to satisfy condition (iii) of Definition 1.2.2, which is a strong basis property for large, in the sense of category, pointsets. So it seems that this problem is related to the problem of producing definable scales for $\neg$IND($\mathbb{R}$) and more complicated pointsets, i.e. the 2nd Victoria Delfino problem (see [9]). Note here the following limitative result: If Γ is a category-adequate Spector pointclass, $\Gamma \subseteq L[\omega^\omega]$ and Det $(P(\omega^\omega) \cap L[\omega^\omega])$ holds, then $\Gamma \subseteq (\boldsymbol{\Sigma}^2_1)^{L[\omega^\omega]}$.

9.2. For each ordinal λ, consider the statement

> D_λ: For every $A \subseteq \lambda^\omega \times \lambda^\omega$ which is definable from a countable sequence of ordinals, the game $G(A; \lambda)$ is determined.

We have established in Corollary 5.4.1 that $D_{\kappa^{\aleph}}$ follows from D_ω, granting AC. Is there some λ, however large, for which D_λ fails? Or, turning things around, does the hypothesis D_λ have any interesting consequences, when asserted for larger and larger λ?

9.3. Does AD imply that ω_1 is κ-supercompact for all $\kappa < \Theta$?

9.4. Does AD imply that all regular cardinals $<\Theta$ are measurable?

9.5. Finally, can one prove Theorem 7.2.1 in Det($\boldsymbol{\Delta}^1_{2n}$) only (when $n \geqslant 1$)? Also, can one prove $\boldsymbol{\Sigma}^1_n$ sets are Ramsey, using Det($\boldsymbol{\Sigma}^1_{n-1}$) only (when $n \geqslant 3$)?

Addendum (September 1980): It has been recently shown in work of Martin, Moschovakis and Steel that much more complicated pointsets than those in **IND**($\mathbb{R}$) admit definable scales. In particular $(\Sigma_1^2)^{L(\omega^\omega)}$ has the scale property and so is a category adequate Spector pointclass. It is assumed here that Det($P(\omega^\omega) \cap L[\omega^\omega]$) holds. (These theorems are still unpublished, but see a brief description in Cabal Seminar 77–79, Lecture Notes in Mathematics, Springer-Verlag.) This extends considerably the range of applicability of the results and methods of the present paper.

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